SCALING RELATIONS FOR THE DETERMINATION OF BEAM OPTICS ERRORS USING RESPONSE MATRIX ANALYSIS

V. Ziemann, The Svedberg Laboratory, S-75121 Uppsala, Sweden

Abstract

Quadrupole gradient errors can be very accurately determined by carefully analysing difference orbits and their deviations from computer-model predictions. This method was successfully applied in synchrotron radiation sources. In order to allow easy before-hand estimates for the application to other accelerators we determine scaling relations for achievable accuracies as a function of BPM resolution and used number of BPM and dipole correctors.

1 INTRODUCTION

In the recent years it has become popular to analyse the linear optics – mostly of synchrotron radiation sources [1, 2, 3, 4] – using response matrix data, i.e. analysing a large number of difference orbits that are obtained by changing a dipole orbit corrector magnet and observing the beam's position change on the BPMs. From the measured response matrix the discrepancies between quadrupole gradients seen by the beam and other hardware related quantities and the corresponding computer-calculated quantities can be determined to a high degree of accuracy.

It is of obvious interest to determine how well the response matrix analysis method is applicable to other accelerators, which have BPMs with much lower resolution than synchrotron radiation sources. Moreover the scaling properties with number of used BPM and correctors is relevant for large machines in order to estimate the trade off when using only a restricted number of BPM or correctors.

The method of response matrix analysis is based on comparing computer-calculated BPM-corrector response elements with measured ones

$$\bar{C}^{ij} = C^{ij} + \sum_k \frac{\partial C^{ij}}{\partial g_k}\delta g_k + C^{ij}\Delta x^i - C^{ij}\Delta y^j \ , \quad (1)$$

where the $\bar{C}^{ij}$ denotes the coefficients measured by observing difference orbits at BPM i generated by changing dipole corrector j, and Δx or Δy are BPM and corrector scale errors, respectively. The quantities we seek to determine are δg_k, the gradient- or other machine-errors such as roll angle or longitudinal position of magnets or BPMs. C^{ij} are model predictions for the response coefficients which in simple cases can be calculated from

$$C^{ij} = \frac{\sqrt{\beta_i\beta_j}}{2\sin\pi\nu}\cos(\phi_i - \phi_j - \pi\nu) \quad (2)$$

where ϕ_i and ϕ_j are the betatron phase at the BPM and corrector, respectively. Equation 1 is a linear equation for the unknown errors $\Delta x^i, \Delta y^j$, and Δg_k which normally is vastly over-determined because we have $N_{\rm bpm}\times N_{\rm cor}$ measured response coefficients per plane and only fit for $N_{\rm bpm} + N_{\rm cor} + N_{\rm quad}$ parameters. In this report the matrix that relates differences between measured and computed response coefficients to the fit parameters will be denoted by A. The system of equations can be solved in the least squares sense [5]. The covariance matrix is given by $(A^tA)^{-1}$ and the accuracies of the fitted parameters are given by the square root of the diagonal elements of the covariance matrix [5]. The finite resolution of each difference orbit measurement is taken into account by dividing each row of A by σ/θ_j, where σ is the finite BPM resolution and θ_j is the deflection affected to the beam by a dipole corrector j. In practice the latter is limited by aperture restrictions or non-linearities from sextupoles.

In this report we discuss the effect of random residual errors on the accuracy to which fit parameters can be determined. Furthermore, we assume that all systematic errors are taken care of by introducing suitable fit parameters. In fact the whole point in doing a global fit to BPM and corrector scales as well as quadrupole gradients and other machine imperfections is to remove these systematic errors in a comprehensive way. The remaining random errors stem from the finite BPM resolution are the subject of the remainder of this paper.

2 BPM AND CORRECTOR ACCURACY

We will first consider the accuracy to which a single corrector scale error can be determined. We assume that the considered corrector j has a scale error y_j which is close to unity. The change of position at BPMs throughout the ring by a presumed corrector kick θ_j is given by

$$z_i = \frac{\sqrt{\beta_i\beta_j}}{2\sin\pi\nu}\cos(\phi_i - \phi_j - \pi\nu)\frac{\theta_j}{y_j} \quad (3)$$

where we have divided by y_j to be consistent with the definition of corrector scale error in eq. 1. Each response coefficient z_j/θ_j is measured with finite accuracy σ/θ_j such that we obtain the following fit equations

$$\begin{pmatrix} \frac{z_1/\theta_j}{\sigma/\theta_j} \\ \frac{z_2/\theta_j}{\sigma/\theta_j} \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{\beta_1\beta_j}\theta_j}{2\sigma\sin\pi\nu}\cos(\phi_1 - \phi_j - \pi\nu) \\ \frac{\sqrt{\beta_2\beta_j}\theta_j}{2\sigma\sin\pi\nu}\cos(\phi_2 - \phi_j - \pi\nu) \\ \vdots \end{pmatrix}\frac{1}{y_j} \ . \quad (4)$$

Now it is easy to calculate the covariance matrix, because the fitting matrix A (including the measurement errors) is just the column vector on the right hand side of eq. 4. Thus we calculate A^tA and get

$$A^tA = \left(\frac{\sqrt{\beta_j}\theta_j}{2\sigma\sin\pi\nu}\right)^2 \sum_{BPM} \beta_i \cos^2(\phi_i - \phi_j - \pi\nu) \ . \quad (5)$$

The sum can be approximated by $N_{\mathrm{bpm}}\beta/2$, where β is the average beta function at places where the BPM are located. The error bar for the inverse scale factor is then given by $\sqrt{(A^tA)^{-1}}$. Note that, since y_j is about unity the error bar for y_j and $1/y_j$ are the same. Moreover we can assume that the correctors are located at places with similar beta functions as the BPM and that all correctors are excited to give the same kick angle θ. We then get for the accuracy to which we can determine the corrector scale error

$$\sigma(y) = 2\sin\pi\nu\frac{\sigma}{\beta\theta}\sqrt{\frac{2}{N_{\mathrm{bpm}}}} \approx \frac{\sigma}{\beta\theta}\frac{1}{\sqrt{N_{\mathrm{bpm}}}} \ . \quad (6)$$

Finally we note that fitting for many correctors simultaneously does not change the argument, because the difference orbits are taken one at a time and the global fit is only done in order to eliminate systematic errors. It does not affect the random errors we are discussing.

Now we consider fitting for the scale error of a fixed BPM from its response to kicks from many different correctors. This problem is the dual problem to the one discussed above as is also clear from eq. 1. Consequently similar arguments apply and we arrive at the following scaling relation for the BPM scale error accuracy

$$\sigma(x) \approx \frac{\sigma}{\beta\theta}\frac{1}{\sqrt{N_{\mathrm{cor}}}} \ . \quad (7)$$

Finally we observe that fitting for gradient errors simultaneously removes systematic errors but does not affect the random errors to which the BPM or corrector scale errors can be determined and equations 6 and 7 hold even when fitting for everything at once.

Note that one global scale factor, the ratio between the average corrector excitation and the average BPM response can not be resolved using this method and the fitting matrix is degenerate. The degeneracy can be resolved, however, by using matrix inversion, based on the Singular Value Decomposition (SVD) Algorithm [5].

3 GRADIENT ACCURACY

We will now discuss the accuracy to which gradient or other parameter-errors can be determined. We will use the phase advance ψ_k in section k of the ring as a model parameter. It is related to integrated gradient errors $\Delta K_1 L$ by $\Delta\psi_k = \beta\Delta K_1 L/4\pi$ where β is the beta function at the place of the gradient error. The relative change of response coefficient is given by the derivative of C^{ij} in eq. 2 with respect to the phase error and the total change of response coefficient is $\partial C^{ij}/\partial\psi_k\,\Delta\psi_k$. Both quantities are readily calculated for a simple FODO model.

We now assume that all systematic BPM and corrector scale errors are accounted for and consider a single quadrupole with gradient error. We use all measured response matrix data to determine its magnitude. The system of equations we have to solve is given by

$$\frac{\bar{C}^{ij} - C^{ij}}{\Sigma^{ij}} = \frac{A_k^{ij}\,\Delta\psi_k}{\Sigma^{ij}} \quad (8)$$

for each of the $N_{\mathrm{bpm}} \times N_{\mathrm{cor}}$ response matrix coefficients. Here Σ^{ij} is the error of a single response coefficient measurement. It can be approximated by

$$\Sigma^{ij} = \sqrt{\sigma^2/\theta^2 + (C^{ij}\sigma(x))^2 + (C^{ij}\sigma(y))^2} \approx \frac{\sigma}{\theta} = \Sigma \ , \quad (9)$$

where $\sigma(x)$ and $\sigma(y)$ are given by eq. 6 and 7, respectively. The approximate equality holds, because we have $C \approx \beta$ and inserting eq. 6 the contribution from the last term under the root is smaller by $1/\sqrt{N_{\mathrm{bpm}}}$ compared to the first. The same is true for the second term. The fitting matrix A is just the column vector given by the derivatives $\partial C^{ij}/\partial\psi_k$. After some algebra we obtain for the inverse of the covariance matrix in the FODO model

$$\frac{A^t}{\Sigma}\frac{A}{\Sigma} = \frac{N_{\mathrm{bpm}}N_{\mathrm{cor}}\beta^2}{16\Sigma^2\sin^2\pi\nu}\frac{1}{2}(1+\cos^2\pi\nu) \ . \quad (10)$$

We can simplify eq. 10 further by ignoring all other terms of order unity and arrive at

$$\sigma(\Delta K_1 L) \approx \frac{\sigma}{\beta^2\theta}\frac{16\pi}{\sqrt{N_{\mathrm{bpm}}N_{\mathrm{cor}}}} \quad (11)$$

when fitting for a single quadrupole gradient error.

We now consider fitting for many quadrupole gradients simultaneously. In that case $(A^t/\Sigma)(A/\Sigma)$ from eq. 10 is a symmetric matrix. If this matrix were diagonal we recovered eq. 10 for the accuracy of each quadrupole. There are, however, correlations in the matrix which can be understood in the following way. Each column in A corresponds to the variation of all response matrix elements to a variation in the associated quadrupole gradient. The matrix A^tA is constructed by calculating the scalar products of every column with all others. The diagonal elements are generated when one column meets its transpose. The off-diagonal elements, on the other hand, describe correlations between gradients. Physically a large correlation implies that gradient errors in corresponding quadrupoles can not be resolved individually. Inspecting A^tA for a simple FODO model we observe that, upon normalization of the diagonal elements to unity, the matrix roughly has the following form

$$\begin{pmatrix} 1 & x & x^2 & x^3 & \dots \\ x & 1 & x & x^2 & \dots \\ x^2 & x & 1 & x & \dots \\ x^3 & x^2 & x & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \ . \quad (12)$$

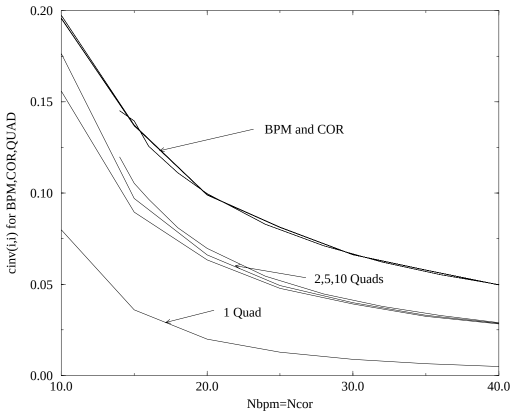

Figure 1: Diagonal elements of the covariance matrix as a function of the used number of BPM and correctors when fitting for different numbers of quadrupoles.

Here x is the normalized correlation between nearby quadrupoles and is on the order of 0.6 to 0.9. Of course this is only a rough approximation. The detailed magnitude of the correlation depends on the beam optical lattice, the chosen quadrupoles, BPM, and correctors. For a rough scaling law, however, it may suffice. Moreover, the normalization is chosen in such a way, as to recover eq. 10 in the limit of only a single fitted gradient.

The matrix in eq. 12 can be inverted to give the numerical correction for the error bars of the fitted gradient errors. One can easily verify that the covariance matrix, which is the inverse of the matrix in eq. 12 is

$$\begin{pmatrix} b^* & a & 0 & 0 & \dots \\ a & b & a & 0 & \dots \\ 0 & a & b & a & \dots \\ 0 & 0 & a & b & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{13}$$

where the coefficients a, b, b^* are given by

$$a = -\frac{x}{1-x^2}\,, \quad b = \frac{1+x^2}{1-x^2}\,, \quad b^* = \frac{1}{1-x^2}\,. \tag{14}$$

Thus we find that the error bars for the gradient scale errors will typically be increased by a factor on the order of $\sqrt{b}$, because the diagonal elements of the covariance matrix are the squares of the error bars of the fitted parameters. Making a conservative pessimistic assumption of $x = 0.9$ we find $b = 9.526$ and the error bars are increased by a factor of about 3, yielding

$$\sigma(\Delta K_1 L) \approx \frac{\sigma}{\beta^2\theta}\frac{48\pi}{\sqrt{N_{\rm bpm}N_{\rm cor}}}\,. \tag{15}$$

We tested the derived relations with the simple FODO model and show the diagonal elements of the covariance matrix $(A^t/\Sigma A/\Sigma)^{-1}$ as a function of BPM and correctors involved in the fit for $N_{\rm bpm} = N_{\rm cor}$ in Fig. 1. The top curve labelled `BPM+COR` shows the accuracy to which the BPM and corrector scales can be fitted. There are three curves superimposed coming from fitting BPM and corrector alone, fitting for BPM, correctors, and 1 quadrupole and the third from fitting BPM, correctors, and 10 quadrupoles. They are rather close, justifying the statements made in section 2 about the independence of fitting the BPM and COR from the quadrupoles. The bottom curves show the accuracy to which the gradient errors can be found if one fits for BPM, correctors and quadrupoles simultaneously, when using 1, 2, 5, or 10 quadrupoles in the fit. We clearly see that the accuracy is about the same if two or more quadrupoles are used in the fit.

4 HORIZONAL AND VERTICAL

Quadrupoles affect the focusing properties in horizontal and vertical plane simultaneously. This makes fitting horizontal and vertical response matrix elements advantageous, because more measurement data are used to determine scale factors and gradient errors.

In a largely uncoupled machine the BPM and corrector scale errors are mostly determined from in-plane data, i.e. the response of horizontal (vertical) BPM to horizontal (vertical) correctors. Thus the argument presented in section 2 applies and the BPM or corrector scale errors are given by eq. 6 and 7.

When determining the gradient errors the number of BPM and corrector used in the fit will simply increase, and eq. 15 needs to be modified by replacing $\sqrt{N_{\rm bpm}N_{\rm cor}}$ by $\sqrt{N_{\rm bpm,x}N_{\rm cor,x} + N_{\rm bpm,y}N_{\rm cor,y}}$ when only in-plane data are used.

The out-of-plane data, i.e. the response of a vertical BPM to horizontal correctors, and vice-versa, yield information about the skew quadrupole gradients and roll misalignment of the quadrupoles, BPM, or correctors. These parameters can be treated similar to gradient errors, except the $N_{\rm bpm,x}N_{\rm cor,y} + N_{\rm bpm,y}N_{\rm cor,x}$ out-of-plane response coefficients are used.

5 REFERENCES

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