MODIFIED MAXWELL-KLIMONTOVICH EQUATION

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Abstract

In some literature, Maxwell-Klimontovich equation (MKE) is utilized in the analysis of the FELs with the premise that the interaction between electrons in a bunch is omitted. However, as is well known, there is cases for FELs that this interaction is considerable, say FELs of Raman type. MKE reckons the reckons the electrons' interaction in, but with the implicit electrostatic approximation which is definitely irrational to the relativistic beam of electrons. As one of our recent works, for arelativistic beam of electron which is in interaction with the radiation photon field, a modified Maxwell-Klimontovich equation (MMKE) is obtained based upon the Liénard-Wiechert potential of a relativistic charged particle. The utilization of the result to the Raman type FELs is under exploiting.

1 INTRODUCTION

K.-J. Kim et al. utilized [1, 2] the Maxwell-Klimontovich equation (MKE) in the analysis of the high gain FELs. Their work are of fundamental importance in the high gain FELs physics. However, in their treatment, the interaction between the electrons in a bunch is omitted. Hence their scheme is applicable only to the Compton FELs. For the Raman FELs, the interaction between the electrons is considerable. MKE as discussed by Y. L. Klimontovich [3] reckons the electrons' interaction in, however the fact that the implicit electrostatic approximation exist therein means that MKE is not suitable for the analysis of the Raman type FELs. Here we establish the modified Maxwell-Klimontovich equation (MMKE), which based upon the Liénard-Wiechert potential of a relativistic charged particle.

2 MAXWELL-KLIMONTOVICH EQUATION

Consider N_e electrons are confined in a space volume V. In the six-dimensional phase space consisting of the position $\vec{r}$ and the velocity $\vec{v}$, each electron has its own trajectory; for i-th electron, denote

$$X_i(t) = [\vec{r}_i(t), \vec{v}_i(t)]. \tag{1}$$

One may take electrons as point particle, thus the microscopic density of the electrons in the phase space is expressed [3] as the summation of the six-dimensional δ-function as

$$N(X;t) = \sum_{i=1}^{N} \delta[X - X_i(t)], \tag{2}$$

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where $X = [\vec{r}, \vec{v}]$. $N(X;t)$ may be called Klimontovich distribution function.

The continuity equation of the distribution function in the phase space yields

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\partial N}{\partial t} + \dot{X} \cdot \frac{\partial N}{\partial X} = 0, \tag{3}$$

or writing in terms of the phase-space coordinates explicitly, we have

$$\frac{\partial N}{\partial t} + \vec{v} \cdot \frac{\partial N}{\partial \vec{r}} + \dot{\vec{v}} \cdot \frac{\partial N}{\partial \vec{v}} = 0, \tag{4}$$

where $\dot{\vec{v}}$ is the acceleration at the phase space point X.

For electrons, the most important acceleration is from the Lorentz force. Thus,

$$\dot{\vec{v}} = \frac{e}{m}[\vec{E}(\vec{r},t) + \vec{v} \times \vec{B}(\vec{r},t)]. \tag{5}$$

The local electric and magnetic field $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ consists of two separate contributions: those from external fields, and those produced by the microscopic fine-grained distribution of the electrons (2),

$$\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}_{ext}(\vec{r},t) + \vec{e}(\vec{r},t), \\ \vec{B}(\vec{r},t) &= \vec{B}_{ext}(\vec{r},t) + \vec{b}(\vec{r},t). \end{aligned} \tag{6}$$

The microscopic fine-grained field resulted from the charge-current of the charged particles. For a given distribution $N(X;t)$, this field is determined by the Maxwell equation. When the electrostatic interaction governs this microscopic fine-grained interaction, one may take the electrostatic approximation and the microscopic field are written as

$$\vec{e}(\vec{r},t) = -\frac{e}{4\pi\varepsilon_0}\frac{\partial}{\partial \vec{r}}\int \frac{N(X';t)}{|\vec{r} - \vec{r}'|}\mathrm{d}X', \quad \vec{b}(\vec{r},t) = 0. \tag{7}$$

Substituting Eq. (7) into Eq. (5) and (4), one may obtain

$$[\frac{\partial}{\partial t} + \mathcal{L}(X) + \int \mathcal{V}(X,X')N(X';t)\mathrm{d}X']N(X;t) = 0. \tag{8}$$

Here, $\mathcal{L}(X)$ is a single-particle operator defined by

$$\mathcal{L}(X) = \vec{v} \cdot \frac{\partial}{\partial \vec{r}} + \frac{q}{m}[\vec{E}_{ext}(\vec{r},t) + \vec{v} \times \vec{B}_{ext}(\vec{r},t)] \cdot \frac{\partial}{\partial \vec{v}}, \tag{9}$$

and $\mathcal{V}(X,X')$ is a two-particle operator arising from the Coulomb interaction which is defined by

$$\mathcal{V}(X,X') = \frac{q^2}{4\pi\varepsilon_0 m}[\frac{\partial}{\partial \vec{r}}\frac{1}{|\vec{r} - \vec{r}'|}] \cdot \frac{\partial}{\partial \vec{v}}. \tag{10}$$

Eq. (8) is called the Maxwell-Klimontovich equation (MKE) [4]; MKE describes the space-time evolution of Klimontovich distribution function as defined by Eq. (2).

3 MODIFIED MAXWELL-KLIMONTOVICH EQUATION

For electron beam in an accelerator, when the energy of the electron are high but not too high, so say have not approached extreme relativistic case. Then the electromagnetic interaction between the electron are both remarkable.

As stated in last section, the microscopic fine-grained field is determined by the Maxwell equations. For a given distribution $N(X;t)$, this microscopic field may be determined by solving the following equations

$$\begin{aligned}
&\nabla \times \vec{e} + \frac{\partial \vec{b}}{\partial t} = 0, \\
&\nabla \times \vec{b} - \frac{1}{c^2}\frac{\partial \vec{e}}{\partial t} = e\mu_0 \int \vec{v} N(X;t) \mathrm{d}\vec{v}, \\
&\nabla \cdot \vec{e} = \frac{e}{\varepsilon_0}\int N(X;t)\mathrm{d}\vec{v}, \\
&\nabla \cdot \vec{b} = 0.
\end{aligned} \tag{11}$$

Suppose that there is no boundary effect, given the distribution function $N(X;t)$, the field can be obtained from the Liénard-Wiechert potential of a relativistic charged particle. For an electron moving with the velocity $\vec{v}(t)$ and following the orbit $\vec{r}'(t)$, the resulting electromagnetic potential reads as following [5]

$$\phi(\vec{r},t) = \frac{e}{4\pi\varepsilon_0(R - \vec{\beta}\cdot\vec{R})}, \quad \vec{A}(\vec{r},t) = \vec{v}\phi(\vec{r},t), \tag{12}$$

wherein $\vec{R} = \vec{r} - \vec{r}'$ and as usual $\vec{\beta} = \vec{v}/c$, the corresponding field reads

$$\begin{aligned}
\vec{E}(\vec{r},t) &= \frac{e}{4\pi\varepsilon_0 c(R - \vec{\beta}\cdot\vec{R})^3} \times \\
&\quad \times\{\frac{c(\vec{R} - R\vec{\beta})}{\gamma^2} + \vec{R}\times[(\vec{R} - R\vec{\beta})\times\dot{\vec{\beta}}]\}, \\
\vec{B}(\vec{r},t) &= \frac{1}{c}\frac{\vec{R}}{R}\times\vec{E}.
\end{aligned} \tag{13}$$

From Eq. (13), the microscopic field are written as

$$\begin{aligned}
\vec{e}(\vec{r},t) &= \frac{e}{4\pi\varepsilon_0 c}\int \mathrm{d}X' \frac{N(X';t)}{(R - \vec{\beta}\cdot\vec{R})^3} \times \\
&\quad \times\{\frac{c(\vec{R} - R\vec{\beta})}{\gamma^2} + \vec{R}\times[(\vec{R} - R\vec{\beta})\times\dot{\vec{\beta}}]\}, \\
\vec{b}(\vec{r},t) &= -\frac{e}{4\pi\varepsilon_0 c^2}\int \mathrm{d}X' \frac{N(X';t)}{(R - \vec{\beta}\cdot\vec{R})^3} \times \\
&\quad \times\{(\frac{c}{\gamma^2} + \vec{R}\cdot\dot{\vec{\beta}})\vec{R}\times\vec{\beta} + (R - \vec{R}\cdot\vec{\beta})\vec{R}\times\dot{\vec{\beta}}\}.
\end{aligned} \tag{14}$$

Substituting Eq. (14) into Eq. (5) and (4), one may obtain after direct computation

$$[\frac{\partial}{\partial t} + \mathcal{L}(X) + \int \mathcal{V}_m(X,X')N(X';t)\mathrm{d}X']N(X;t) = 0, \tag{15}$$

with single-particle operator $\mathcal{L}(X)$ defined by Eq. (9) just the same as that of the MKE, while the two-particle operator $\mathcal{V}_m(X,X')$ defined as

$$\begin{aligned}
&\mathcal{V}_m(X,X') \\
&= \frac{e}{4\pi\varepsilon_0 c(R - \vec{\beta}\cdot\vec{R})^3} \times \\
&\quad \times\{[\frac{c}{\gamma^4} + \frac{\vec{R}\cdot\dot{\vec{\beta}}}{\gamma^2} - (R - \vec{\beta}\cdot\vec{R})(\vec{\beta}\cdot\dot{\vec{\beta}})]\vec{R} \\
&\quad -(R - \vec{\beta}\cdot\vec{R})^2\dot{\vec{\beta}} + (\frac{c}{\gamma^2} + \vec{R}\cdot\dot{\vec{\beta}})(\vec{R}\cdot\vec{\beta})\vec{\beta}\}.
\end{aligned}$$

One may name Eq.(15) as modified Maxwell-Klimontovich equation (MMKE); MMKE describes the space-time evolution of Klimontovich distribution function as defined by Eq. (2).

4 THE MOMENT'S EQUATIONS

One will find that these equations for MMKE are similar to that of MKE, as is done in [3]. One now introduced an averaging process based upon the Liouville distribution over the $6N_e$-dimensional phase space (the Γ space). The microscopic state of the system is expressed in the Γ space by a point

$$\{X_i\} = (X_1, X_2, \cdots, X_n),$$

which may be called a system point. Following a formal procedure of the ensemble theory in statistical mechanics, we may imagine $\mathcal{N}$ replicas which are macroscopically identical to the system under consideration; the number $\mathcal{N}$ may be chosen as large as we like, so that we may let it approach infinity whenever convenient. These $\mathcal{N}$ replicas are generally characterized by different microscopic configurations; the system points are scattered over the Γ space. One may define the Liouville distribution function in the Γ space; let the number of the total system point $\mathcal{N}$ is large enough, and the distribution function $D(\{X_i\};t)$ defined as the system point density divided by $\mathcal{N}$. Hence it satisfied the normalization condition $\int D(\{X_i\};t)\mathrm{d}\{X_i\} = 1$. The $\mathcal{N}$ system point distributed in the Γ space apparently do not interacted with each other; they behave like an ideal gas. Consequently the distribution function $D(\{X_i\};t)$ satisfies a grand continuity equation of Liouville type:

$$\frac{\partial D(\{X_i\};t)}{\partial t} + \{\dot{X}_i\}\cdot\frac{\partial D(\{X_i\};t)}{\partial\{X_i\}} = 0. \tag{16}$$

Along a trajectory in the phase space, the distribution is conserved.

With the aid of the Liouville distribution, one may carry out a statistical averaging of the a fine-grained quantity $A(X, X', \cdots; \{X_i(t)\})$, defined at a set of points $(X, X', \cdots)$ in the six-dimensional phase space in the following way

$$\begin{aligned}
&\langle A(X,X',\cdots;t)\rangle \\
&= \int \mathrm{d}\{X_i\} D(\{X_i(t)\};t)\times A(X,X',\cdots;\{X_i(t)\}).
\end{aligned} \tag{17}$$

Due to $D(\{X_i\};t)\mathrm{d}\{X_i(t)\} = D(\{X_i\};0)\mathrm{d}\{X_i(0)\}$, the conservation condition, this average can be equivalently be transformed into an average over the initial distribution, so that

$$\langle A(X,X',\cdots;t)\rangle = \int \mathrm{d}\{X_i(0)\}D(\{X_i(0)\};0)\times \times A(X,X',\cdots;\{X_i(\{X_i(0)\};t)\}),$$

where $\{X_i(\{X_i(0)\};t)\}$ represents the coordinates of the system points in the Γ space at t under the condition that it was located at $\{X_i(0)\}$ when $t=0$.

One is now ready to perform the averaging to the MMKE with respect to distribution function $D(\{X_i(t)\};t)$. One may first check that

$$\langle N(X;t)\rangle = \int \mathrm{d}\{X_i(0)\}D(\{X_i(0)\};0)\sum_{i=1}^{N}\delta[X-X_i(t)] = f_1(X;t), \tag{18}$$

and

$$\begin{aligned}&\langle N(X;t)N(X';t)\rangle \\ &= \int \mathrm{d}\{X_i(0)\}D(\{X_i(0)\};0)\times \\ &\quad \times\sum_{i=1}^{N}\delta[X-X_i(t)]\delta[X'-X_i(t)] \\ &= \delta[X-X']f_1(X;t)+f_2(X,X';t),\end{aligned} \tag{19}$$

where $f_1(X;t)$ and $f_2(X,X';t)$ is called 1-st and 2-nd moment of the Klimontovich distribution function. One may adopt the shorthand notation for $X, X', X'', \cdots$ as 1, 2, 3, $\cdots$. When the averaging is performed to the Eq. (15), the result is expressed as following

$$[\frac{\partial}{\partial t}+\mathcal{L}(1)]f_1(1;t) = \int \mathcal{V}(1,2)f_2(1,2;t)\mathrm{d}2. \tag{20}$$

One may likewise start from an equation

$$\begin{aligned}&[\frac{\partial}{\partial t}+\mathcal{L}(1)+\mathcal{L}(2)]N(1;t)N(2;t) \\ &= \int [\mathcal{V}(1,3)+\mathcal{V}(2,3)]N(1;t)N(2;t)N(3;t)\mathrm{d}3,\end{aligned}$$

which may be obtained from a combination of MMKE. Upon averaging this equation with respect to the Liouville distribution, one may get

$$\begin{aligned}&[\frac{\partial}{\partial t}+\mathcal{L}(1)+\mathcal{L}(2)-\mathcal{V}(1,2)-\mathcal{V}(2,1)]f_2(1,2;t) \\ &= \int [\mathcal{V}(1,3)+\mathcal{V}(2,3)]f_3(1,2,3;t)\mathrm{d}3,\end{aligned}$$

where $f_3(1,2,3;t)$ is the 3-rd moment.

Quite similarly, one can consider an equation for a product of an arbitrary number of the Klimontovich functions and carry a statistical average of this function. One may thus obtain a hierarchy equation of the moments, which may be expressed in the following way

$$\begin{aligned}&[\frac{\partial}{\partial t}+\sum_{i=1}^{s}\mathcal{L}(i)-\sum_{i\neq j}^{s}\mathcal{V}(i,j)]f_s(1,2,\cdots,s;t) \\ &= \sum_{i=1}^{s}\int \mathcal{V}(i,s{+}1)f_{s+1}(1,2,\cdots,s;t)\mathrm{d}(s{+}1).\end{aligned} \tag{21}$$

5 CONCLUSION REMARKS

Here we establish the modified Maxwell-Klimontovich equation (MMKE), which based upon the Liénard-Wiechert potential of a relativistic charged particle. Its application to FELs of Raman type is under exploring.

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