NEW POTENTIAL FUNCTION FOR RFQ ACCELERATOR CELLS

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Abstract

The performance of a rf quadrupole (RFQ) accelerator design is assessed (in advance of construction) by computer tracking of particles, from one cell to the next, through a model of the electric fields that transport the beam. Most computer models are based on the potential of Kapchinskii and Tepliakov[1] which assumes a perfectly periodic structure, whereas an RFQ accelerator is only a quasi-periodic device; the cells are typically adapted to accept a d.c. beam, bunch, and then accelerate. A short coming of the two-term potential is that there are only two free parameters; this may lead to discontinuity at the cell boundary. We introduce a new potential and unit cell, with independently variable bore radius and minimum and maximum radii to pole tip, that may better approximate the real-world fields. Other specialist potentials are also examined.

1 INTRODUCTION

The concept of a *unit-cell* whose properties are completely independent of its neighbours is useful because it allows one to design cells for specific functions; for example a pure focusing cell, or a cell optimized for acceleration. Of course, the concept is an approximation and neighbour cells mutually influence one another. Let $\beta = v/c$ be the relativistic factor and λ be the wavelength of the RF; and define $k = 2\pi/(\beta\lambda)$. We adopt cylindrical polar coordinates r, θ, z. Cells are modelled by the 2-term potential:

$$\Phi(r,\theta,z) = A_{01} r^2 \cos 2\theta + A_{10} \cos(kz) I_0(kr) \ . \quad (1)$$

Here J_n is the common Bessel function of order n and $J_n(iz) = (i)^n I_n(z)$ where $i = \sqrt{-1}$. Because there are only two free parameters A_{01}, A_{10}, specifying the radii to pole-tip as ($r = a_1, \theta = 0, z = 0$) and ($r = a_2, \theta = \pi/2, z = 0$) completely specifies a cell. However, in this case, the radius to pole-tip at the ends of the cell $[z = \pm\pi/(2k)]$ is an uncontrolled variable.

Communities of cells are organized into sections with dedicated tasks[2, 3], such as "mostly focusing" or "mostly accelerating"; e.g. the RMS, the buncher, the accelerator. Clearly, a potential function capable of specifying a group of cells according to some long range modulation of cell parameters would be useful; one can consider this as an over-arching curvature (in the r, z-plane) of the vanes either away or toward the optic axis.

To summarize, this paper has two strands: to search for a more versatile accelerator single cell; and to find potentials for groups of cells. All potential functions, Φ, discussed here satisfy Laplace's equation $\nabla^2\Phi = 0$.

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2 UNIT CELL

By convention, the unit cell is one half period of the modulation. The periodic structure is generated by successive application of the operations: a translation of $z = \pi/k$ and a rotation of $\theta = \pi/2$. Often, in text books, the unit cell is shown as extending from $z = 0$ to $z = \pi/k$, locations corresponding to the maximum and minimum of the transverse modulation for one (diametrically opposite) pair of vanes. Though this is acceptable when neighbour cells have identical values of A_{01}, A_{10}, this is unfortunate when they have *different* parameters because it leads to the conclusion that the vane-shapes become discontinuous at the cell boundaries. So widespread and so unfortunate is this choice of a unit cell that we have a whole page discussion and recipe by Puglisi[4](pg. 715) for the connection of two adjacent cells (with different parameters) that ends with the comment: "From the above arguments it is evident that there is no continuity between adjacent cells and that the previous procedure should be modified..."

Rather, the unit cell should be taken as extending from $z = -\pi/(2k)$ to $z = +\pi/(2k)$ whence adopting constant characteristic bore radius r_0 will automatically enforce continuity. This definition may eliminate the need for the transition cells introduced by Crandall[5]. To summarize, judicious choice of the cell boundaries moves a possible discontinuity from *function value* to *function derivative*. However, there is the possible disadvantage that the new cell has two half accelerating gaps (one at either end) in place of a central single gap.

2.1 Field Discontinuity

The above prescription for a symmetric unit cell is not sufficient to avoid field discontinuity in a non-periodic structure. Suppose that for the first in a pair of cells:

$$\Phi(r_0, 0, \pm\pi/2k) = \Phi(a_1, 0, 0) = V \ , \quad (2)$$

$$A_{01} = V/r_0^2 \ , \qquad A_{10} = A_{01}(r_0^2 - a_1^2)/I_0(ka_1) \ . \quad (3)$$

The second cell is obtained by substituting maximum radius to pole-tip b_1 in place of a_1 and wave-number l in place of k. Comparing values at the cell boundary $z = \pi/2k$ one obtains $\mathbf{E}(r,\theta,+\pi/2k) + \mathbf{E}(r,\theta+\pi/2,-\pi/2l)$

$$= \mathbf{e}_z[kA_{10}(k,a_1)I_0(kr) - lA_{10}(l,b1)I_0(lr)] \ . \quad (4)$$

Hence the vane-profile and potential are continuous, but not the longitudinal component of the electric field model. The field becomes continuous *on-axis* if

$$[k(a_1^2 - r_0^2)/I_0(ka_1)] = [l(b_1^2 - r_0^2)/I_0(lb_1)] \ . \quad (5)$$

This may be viewed as the *adiabaticity condition* which if implemented will make the simulations better approximate the real world. Unfortunately, complete self-consistency

between k, a_1, l, b_1 and the acceleration in those cells is not always possible.

There is a similar problem that occurs in the implementation of radial matching sections (RMS) in particle tracking programs where the RMS is modelled as a region with ostensibly no modulation, but a different r_0 for each cell. N.B. PARMTEQ, does attempt to make some correction for this inconsistency.

3 FOCUSING SECTIONS

3.1 Radial Matching Section

The radial matching section[6] is a good example of a long term variation of parameters. Tokuda[7] gave the following potential function $\Phi = A_{11}\cos(lz)\cos(2\theta)I_2(lr)$ with $A_{11} = V/I_2(lr_0)$ and $z \geq 0$. The end wall is placed at $lz = \pi/2$, a multiple of $\beta\lambda/2$. The potential is discontinuous when this RMS is mated with a conventional A_{01} quadrupole cell at $z = 0$. Adherence to the Tokuda vane profile is unnecessary, and modelling codes typically have the option to input a series of cells with $A_{10} = 0$ but varying A_{01}. We consider $\Phi = r^2\cos 2\theta[c_1 + c_2 z]$. One may specify the constants globally as $c_1 = V/r_0^2$ and $c_2 = -c_1 \times (2/\pi)l$. Or specify them locally through the radius to pole-tip at the cell boundaries $\Phi(r_1, 0, -\pi/2k) = \Phi(r_2, 0, +\pi/2k) = V$, then the vanes and potential are continuous and the constants are given by:

$$c_1 = \frac{V}{2}\left[\frac{1}{r_2^2} + \frac{1}{r_1^2}\right] \qquad c_2 = \frac{kV}{\pi}\left[\frac{1}{r_2^2} - \frac{1}{r_1^2}\right] . \quad (6)$$

The longitudinal electric field component is continuous at the boundary between cells with bore radii r_1, r_2 and r_2, r_3 iff $r_3 = r_1 r_2/\sqrt{2r_1^2 - r_2^2}$. Evidently using the radius ratio $\sqrt{2}$ in the penultimate cells will give a final cell in which the radius to pole tip becomes infinite.

In addition to the horn-shape of the RMS, it is possible to consider pinch-shape vanes such as that given by $\Phi = V\cosh(lz)\cos(2\theta)J_2(lr)/J_2(lr_0)$ and these may have applications in trapping.

3.2 Split Tunes

The conventional A_{01} AG-focusing makes the betatron tunes equal and is naturally suited to beams of equal horizontal and vertical emittance. The potential $\Phi = A_{11}I_2(kr)\cos 2\theta\cos kz$ may be added to the usual A_{01} term to add some constant gradient quadrupole focusing that will split the tunes. This provides extra focusing in the plane for which the emittance is larger; but at the cost of introducing off-axis accelerations.

3.3 Transition Cells

Our use of *transition* differs from Crandall. The need may arise to join focusing cells with differing characteristic bore radii. Two candidate forms for this function are:

$$\Phi = J_2(mr)\cos 2\theta[A_{11}\cosh mz + B_{11}\sinh mz] , \quad (7)$$
$$\Phi = I_2(mr)\cos 2\theta[A_{11}\cos mz + B_{11}\sin mz] . \quad (8)$$

Both expressions reduce to $A_{01}\cos(2\theta)(mr)^2/8$ in the limit $m \to 0$, and so we recover the pure quadrupole; satisfying boundary conditions will give $A_{01} = 8V/(mr_0)^2$ and so Φ remains non-zero. In the case of cells which are symmetrical about their center, $B_{11} = 0$, it is easy to decide which form to take. If the vane-shapes curve back toward the optic axis, then we should take the hyperbolic function $\cosh(mz)$; and if the vane-shapes curve longitudinally away from the axis then we take the trigonometric function $\cos(mz)$. In the case that the longitudinal vane profile is asymmetric, the situation is less intuitive and some analysis is required. Suppose we are given the geometric constraint

$$\Phi(r_0, 0, 0) = \Phi(r_1, 0, +b) = \Phi(r_2, 0, -b) = V . \quad (9)$$

$$\text{Form the quantity} \quad T = \frac{r_0^2}{2}\left[\frac{1}{r_2^2} + \frac{1}{r_1^2}\right] . \quad (10)$$

If $T < 1$ then use trigonometric functions; if $T > 1$ then adopt hyperbolic functions; and if $T = 1$ the vane profile is flat ($r_0 = r_1 = r_2$) and $m = 0$. If $b = \pi/2k$ then the transition is effected in one cell.

Locally convex vane profile, $T < 1$

$$A_{11} = V/I_2(mr_0) , \quad (11)$$
$$B_{11} = A_{11}\frac{[I_2(mr_2) - I_2(mr_1)]}{[I_2(mr_1) + I_2(mr_2)]}\cot(bm) . \quad (12)$$

The cell coefficients are given above. The curvature parameter m is given by the iterative equation:

$$bm = \arccos\left[\frac{I_2(mr_0)}{2}\left[\frac{1}{I_2(mr_2)} + \frac{1}{I_2(mr_1)}\right]\right] . \quad (13)$$

In most cases an acceptable approximation is obtained if one substitutes into the right hand side of this equation the value $bm = \arccos(T)$.

Locally concave vane profile, $T > 1$ The coefficients A_{11}, B_{11} are given as above but with the replacements $I_2(z) \Rightarrow J_2(z)$ made throughout, and $\coth(x)$ replaces $\cot(x)$ and $\text{arccosh}(x)$ replaces $\arccos(x)$.

4 RFQ ACCELERATOR CELLS

It is not topologically possible to manufacture vanes which conform exactly to the hyperbolic-and-Bessel dependence. As a consequence, the fields in the cell interior differ from the ideal and the focusing or acceleration may depart from design values. Typically parameters, for the true cell geometry, are adjusted until sufficient acceleration is recovered. In this process, the cherished principle of "constant r_0" is typically broken. Ideally, for reasons of versatility, it is clearly desirable to have a new potential function and a new type of unit cell that allows the modulation depth (determined from a_1, a_2) to be specified independently and yet maintain the same bore radius r_0. Note, we are not suggesting to re-shape vane-profiles, rather (for the purpose of particle-tracking) we wish to find a potential function that more closely adheres to the "as cut" profiles.

4.1 Asymmetric Cell

In addition to the symmetrical unit-cell with equal bore radius, r_0, at the cell ends, a specialist application may require asymmetric cells as transitions between accelerator sections with differing characteristic bore radii r_1, r_2. The minimalist potential-function consistent with these requirements is to supplement the $A_{10}I_0(kr)\cos(kz)$ acceleration term with either of the focusing potentials (7) or (8) as a replacement for the usual $A_{01}r^2\cos 2\theta$ term.

$$\Phi(a_1, 0, 0) = +V\ , \qquad \Phi(a_2, \pi/2, 0) = -V\ , \qquad (14)$$

$$\Phi(r_1, 0, -\pi/2k) = \Phi(r_2, 0, +\pi/2k) = +V\ . \qquad (15)$$

These geometric constraints above lead to simultaneous equations for the cell coefficients: (14) give A_{11}, A_{10} while (15) give B_{11}, m. We introduce T, a measure of the longitudinal curvature, so as to determine whether trigonometric or hyperbolic functions of mz are required.

$$T = \frac{[2(a_1^2 + a_2^2) + (a_1 a_2 k)^2]}{[8 + k^2(a_1^2 + a_2^2)]}\left[\frac{1}{r_2^2} + \frac{1}{r_1^2}\right]\ . \qquad (16)$$

If $T < 1$ then we use form (8) $\cos(mz)$ and if $T > 1$, we use (7) $\cosh(mz)$. If $T = 1$ then $m = 0$ and both forms reduce to the classic A_{01} quadrupole potential. Results for the asymmetric cell reduce to the symmetric form under the substitution $r_1 = r_2 = r_0$.

Locally concave bore $T > 1$

$$A_{11} = V[I_0(ka_1) + I_0(ka_2)]/D(a_1, a_2, k, m)\ , \qquad (17)$$

$$A_{10} = V[J_2(ma_2) - J_2(ma_1)]/D(a_1, a_2, k, m)\ , \qquad (18)$$

$$B_{11} = A_{11}\frac{[I_2(mr_1) - I_2(mr_2)]}{[J_2(mr_1) + J_2(mr_2)]}\coth\left(\frac{m\pi}{2k}\right)\ . \qquad (19)$$

The cell coefficients are given above. Here denominator $D = I_0(ka_2)J_2(ma_1) + I_0(ka_1)J_2(ma_2)$. The curvature parameter is given by: $m\pi/(2k) =$

$$\mathrm{arccosh}\left[\frac{D(a1, a2, k, m)}{2[I_0(ka_1) + I_0(ka_2)]}\left[\frac{1}{J_2(mr_1)} + \frac{1}{J_2(mr_2)}\right]\right]\ . \qquad (20)$$

This non-linear equation must be solved self-consistently for m. Unfortunately, this equation is singular at $m = 0$ and so recursive solution must begin at the value given by:

$$m\frac{\pi}{k} = 2\mathrm{arccosh}\left[\frac{[a_2^2 I_0(ka_1) + a_1^2 I_0(ka_2)]}{4[I_0(ka_1) + I_0(ka_2)]}\left[\frac{1}{r_2^2} + \frac{1}{r_1^2}\right]\right]\ . \qquad (21)$$

Often sufficient accuracy is obtained if the m-value from (21) is substituted in the right side of (20). Alternatively, one may start the recursion from $m(\pi/k) = 2\mathrm{arccosh}(T)$.

Locally convex bore $T < 1$ The coefficients are given as above but with the replacement $J_2(x) \Rightarrow I_2(x)$ made throughout. The hyperbolic terms $I_0(x)$ *do not* change since they come from the acceleration term A_{10}. For the curvature parameter, $\arccos(x)$ replaces $\mathrm{arccosh}(x)$.

4.2 Longterm Modulation of Accelerator

One may wonder if there is a potential function which specifies a group of accelerator cells according to some long range modulation of cell parameters.
We introduce the following quantities: $i^2 = -1$, $j^2 = +1$, $I^2 = -1$. Then $e^{jz} = (\cosh z + j\sinh z)$. The potential $\Phi = e^{ikz}e^{jmz}e^{In\theta}J_n[(ik + jm)z]$ satisfies Laplace's equation as does each of its 8 components $(1, i, j, ij)$ and $I \times (1, i, j, ij)$. Reversing the sign of exponents i, j simultaneously and or I leaves the Bessel function unchanged. When n is even the simplest accelerator-like combination of terms with a product of modulations is

$$\Phi = [B_n^1 \cos kz \cosh mz - B_n^{ij} \sin kz \sinh mz]\cos n\theta\ , \qquad (22)$$

where B_n^1 and B_n^{ij} are the real and ij components of $J_n[(ik + jm)z]$. Let $\kappa = \sqrt{k^2 - m^2}$ and suppose $m \ll k$. $J_n = B_n^1 + ijB_n^{ij} \approx (i)^n[I_n(\kappa r) - ijI_n'(\kappa r)mkr/\kappa]$ when n is even. Unfortunately, the asymmetry about $z = 0$ makes (22) not very useful. The superposition of functions required to give a pure $\cos kz \cosh mz$ longitudinal dependence leads to the inclusion of a complicated Bessel function dependence which conflicts with our goal of short, simple expressions to facilitate analytic calculations.

5 SUMMARY

Based on the idea of longitudinal curvature of the vanes we have presented a variety of potential functions which may be broadly categorized as *horn* or *pinch*. These may be exploited to give a versatile unit-cell with independent modulation depth and bore radii. Applying the idea of curvature to manipulate a group of cells, though trivial for focusing sections, does not lead to simple expressions for accelerator sections.

6 REFERENCES

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