THEORETICAL ANALYSIS OF A REAL-LIFE RFQ USING A 4-WIRE LINE MODEL AND THE THEORY OF DIFFERENTIAL OPERATORS

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Abstract

A comprehensive mathematical model is needed to interpret the fields measured in a real-life RFQ (radio-frequency quadrupole) in terms of mechanical errors, end detuning or tuners commands. The central region of the RFQ is modeled as a loaded 4-wire transmission line. The inter-vane voltages are solution of a vector differential equation, which is recast into an eigen-value problem, leading to a bi-orthonormal base and associated inner product. The real-life RFQ is viewed as a perturbation of a perfect one, and measured fields may be expanded on the eigen-basis. Field perturbations are then transformed into line perturbations, as capacitances errors, or tuner 'inverse inductances'. A multiple excitation technique is also used for end/coupling cells tuning.

1 INTRODUCTION

A high current, high power RFQ is usually tuned by more than 100 'tuning devices', like slugs, plates thickness etc. An efficient tuning algorithm should (i) rely on an accurate electromagnetic model which saves the effects of real-life perturbations, and (ii) be fast enough to allow for a few tuning iterations in a reasonable amount of time. An equivalent circuit in the form of a 4-wire transmission line is a good candidate, since it preserves the 3-D features of the problem, including boundaries, and may be analyzed with the (powerful) theory of linear differential operators.

2 THE 4-WIRE LINE MODEL

2.1 The (non-dispersive) TEM 4-wire line

The current section of a 4-vanes RFQ is a bounded, simply connected domain. The only possible waveguides modes are either TE (transverse-electric) or TM (transverse-magnetic). The general solution of Maxwell equations for the RFQ TE modes is well-known [1], [2]:

$$\overline{H}_t \sim \overline{\nabla}_t\psi\ I(z)\ ,\ \overline{E}_t \sim \hat{z}\times\overline{\nabla}_t\psi\ V(z)\ ,\ H_z \sim \psi\ V(z)\ ,$$

where the subscript t denotes transverse fields, $\hat{z}$ is the axial unit vector, γ the wave-number, and where the modal function $\psi(x, y)$ satisfies:

$$\overline{\nabla}_t \bullet \overline{\nabla}_t\psi + k_m^2 . \psi = 0 \quad \text{with} \quad \partial\psi/\partial n\big|_\Gamma = 0\ .$$

The Neumann-Laplacian operator is negative self-adjoint, with a discrete spectrum. The eigen-functions are the usual wave-guide modes, and the eigen-values k_m are the cut-off wave-numbers. A waveguide with a multiply connected section Ω, may support TEM (transverse-electric-magnetic) modes:

$$\overline{H}_t \sim \overline{\nabla}_t\varphi_j\ I(z)\ ,\ \overline{E}_t \sim \hat{z}\times\overline{\nabla}_t\varphi_j\ V(z)\ ,\ \overline{\nabla}_t \bullet \overline{\nabla}_t\varphi_j = 0\ ,$$

$$\text{with: } \partial\varphi_j/\partial n\big|_\Gamma = 0\ ,\ \varphi_j\big|_{\Sigma_i} = \delta_{ij}\ ,\ \partial\varphi_j/\partial n\big|_{\Sigma_i} = 0\ .$$

Here, Σ_i , i = 1 ... N are cuts of Ω such that the domain $\Omega \backslash \cup\ \Sigma_i$ be simply connected [3]. In the axial region of the RFQ where ψ is numerically negligible, the TEM fields give a good image of the TE fields. The RFQ is thus represented by a system of 4 wires (the electrode tips or the rods) supporting a TEM mode. Introducing the following 3-vectors and 3x3 arrays (for a 4-wire system in unbounded space), with conductor #4 chosen as common reference:

$\overline{u}$ = voltage, $\overline{i}$ = current, $\overline{q}$ = charge per unit length,

$\overline{\phi}$ = density of magnetic flux per unit length,

$\overline{\overline{C}}$ = 'capacitance per unit length' array,

$\overline{\overline{L'}}$ = 'inductance per unit length' array,

the line equations are: $\overline{q} = \overline{\overline{C}}\,\overline{u}$ and $\overline{\phi} = \overline{\overline{L'}}\,\overline{i}$.

The capacitance is supposed to be known from the RFQ design (through SUPERFISH computations for instance), and the inductance array is computed from the fundamental property [4] of TEM lines (c = velocity of waves in surrounding medium):

$$c^2\,\overline{\overline{L'}}\,\overline{\overline{C}} = \overline{\overline{1}}\ .$$

2.2. The dispersive 4-wire line

The longitudinal magnetic field of the RFQ waveguide acts as a current load for each pair of adjacent conductors, and is introduced in the model as a parallel 'inverse inductance per unit length' array $\overline{\overline{L}}$. The full model (fig.1) is described by the set of equations:

$$\partial_z\overline{u} = -j\omega\,\overline{\overline{L'}}\,\overline{i} \qquad \& \qquad \partial_z\overline{i} = -\left[\ j\omega\,\overline{\overline{C}} + \frac{1}{j\omega}\,\overline{\overline{L}}\ \right]\overline{u}\ .$$

Note that all arrays involved are *full* and *symmetric*.

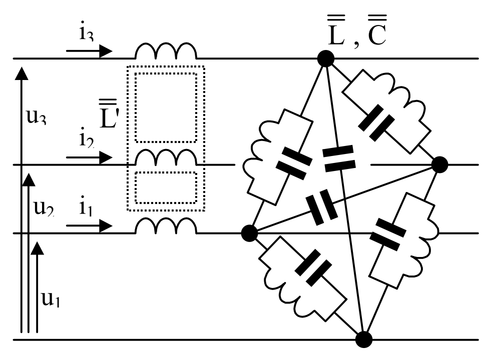

Fig.1 – *The 4-wire line equivalent circuit*

3 SPECTRAL THEORY

3.1 Differential equation

The two line equations may be combined to form a second order differential equation in $\overline{u}$:

$$\partial_z^2\overline{u} - \frac{1}{c^2}\,\overline{\overline{C}}\,\overline{\overline{L}}\,\overline{u} = -\frac{\omega^2}{c^2}\,\overline{u}$$

(neglecting all derivatives of capacitance and inductance). Applying the physically meaningful base transform:

$$\overline{U} = \begin{vmatrix} U_Q \\ U_S \\ U_T \end{vmatrix} = \begin{vmatrix} +1 & -1 & +1 \\ +1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} \\ +1/\sqrt{2} & +1/\sqrt{2} & -1/\sqrt{2} \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix} \equiv \overline{\overline{S}}^{-1} \, \overline{u}$$

the equation becomes:

$$\partial_z^2 \overline{U} - \overline{\overline{A}}\,\overline{U} = -\omega^2/c^2\,\overline{U} \quad \text{with} \quad \overline{\overline{A}} = c^{-2}\,\overline{\overline{S}}^{-1}\,\overline{\overline{C}}^{-1}\,\overline{\overline{L}}\,\overline{\overline{S}}\,,$$

and $\overline{\overline{A}}$ is *diagonal for a perfectly symmetric* (only) *RFQ*. The Q axis corresponds to the desired quadrupolar component; the S and T axes are dipolar components. Denoting U_i the potential of conductor #i+1 with respect to conductor #i:

$$\begin{vmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{vmatrix} = \begin{vmatrix} -1/2 & -1/\sqrt{2} & 0 \\ +1/2 & 0 & -1/\sqrt{2} \\ -1/2 & +1/\sqrt{2} & 0 \\ +1/2 & 0 & +1/\sqrt{2} \end{vmatrix} \begin{vmatrix} U_Q \\ U_S \\ U_T \end{vmatrix}.$$

3.2 Differential operator

The 3-vector $\overline{U}$ is defined on a support of the form: $\Omega =]a\,, c_1[\cup]c_1\,, c_2[\cdots \cup]c_n\,, b[$, where a and b are the locations of the RFQ ends #1 and #2, and the c_i are the locations of the *coupling cells* (if any) connecting adjacent RFQ segments. Solutions of the differential equation are to be found in the space of square integrable 3-functions over Ω, $L^2(\Omega)^3$, and more precisely in the Sobolev space:

$$H^2(\Omega) = \left\{ \overline{f} \in L^2(\Omega)^3\,, \partial^2\overline{f} \in L^2(\Omega)^3 \right\}.$$

Let's define the inner product in $H^2(\Omega)$:

$$\langle \overline{f}_1, \overline{f}_2 \rangle = \int_\Omega \overline{f}_1^{\,*} \bullet \overline{f}_2 \, d\Omega \qquad \text{(sesquilinear form).}$$

Then the operator $\overline{\overline{M}} = \overline{\overline{1}}\,\partial_z^2 - \overline{\overline{A}}$ can be shown to be self-adjoint on sub-spaces of $H^2(\Omega)$, defined by any combination of the following boundary conditions, and with $\overline{\overline{A}}$ hermitian in all cases:

at end #1:

$\partial_z \overline{f}\big|_a = \overline{\overline{s}}_a\,\overline{f}(a)$, with $\overline{\overline{s}}_a$ hermitian, *or* $\overline{f}(a) = \overline{0}$,

at end #2:

$\partial_z \overline{f}\big|_b = \overline{\overline{s}}_b\,\overline{f}(b)$, with $\overline{\overline{s}}_b$ hermitian, *or* $\overline{f}(b) = \overline{0}$,

at any c_i:

$$\begin{vmatrix} U_X(c_i^-) \\ \partial_z U_X(c_i^-) \end{vmatrix} = \overline{\overline{d}} \begin{vmatrix} U_X(c_i^+) \\ \partial_z U_X(c_i^+) \end{vmatrix}, \ X = Q, S \text{ or } T,$$

with $\overline{\overline{d}} = \overline{\overline{d}}_0\,e^{j\varphi}$, φ arbitrary, $\overline{\overline{d}}_0$ real, and $\det(\overline{\overline{d}}_0) = 1$. The above-mentioned sub-spaces are Hilbert spaces, with the norm defined by the inner product. Eigen-values are all real, and eigen-function are also real functions up to a complex multiplicative constant. The set of eigen-functions is an orthogonal (orthonormal after normalization) basis for the space, and may be split into three subsets $\overline{V}_{Qi} = \hat{Q}\,V_{Qi}$, $\overline{V}_{Sj} = \hat{S}\,V_{Sj}$, $\overline{V}_{Tk} = \hat{T}\,V_{Tk}$ corresponding to the decoupling of Q, S and T in the perfectly symmetric case. Note that for a perfectly symmetrical, perfectly loaded RFQ, the S and T eigen-frequencies are degenerated, and the corresponding axes form a base of this 2-D eigen-space. Another interesting property is that the density of the eigen-values square roots tends to a constant for high order [5]:

$$\sqrt{\lambda_k} = k.\pi/(b-a) + O(1/k) \quad \text{i.e.} \quad \omega_k \# kc\pi/(b-a),$$

corresponding to a phase velocity going to c when ω goes to infinity.

4 PERTURBATION ANALYSIS

4.1 Perturbation of the differential operator

Let $\Delta\overline{\overline{M}}$ be a (small) perturbation of the operator. It can be shown that any perturbed eigen-function, Q_{i0} for instance, can be expanded at first order as:

$$\overline{V}'_{Qi0} = \overline{V}_{Qi0} + \sum_{i \neq i0} \frac{\langle \overline{V}_{Qi}\,, \Delta\overline{\overline{M}}\,\overline{V}_{Qi0} \rangle}{-(\omega_{Qi0}^2 - \omega_{Qi}^2)/c^2}\,\overline{V}_{Qi} + \sum_j \frac{\langle \overline{V}_{Sj}\,, \Delta\overline{\overline{M}}\,\overline{V}_{Qi0} \rangle}{-(\omega_{Qi0}^2 - \omega_{Sj}^2)/c^2}\,\overline{V}_{Sj} + \sum_k \frac{\langle \overline{V}_{Tk}\,, \Delta\overline{\overline{M}}\,\overline{V}_{Qi0} \rangle}{-(\omega_{Qi0}^2 - \omega_{Tk}^2)/c^2}\,\overline{V}_{Tk}$$

and the perturbed eigen-value as:

$$-(\omega'^2_{Qi0} - \omega^2_{Qi0})/c^2 = \langle \overline{V}_{Qi0}\,, \Delta\overline{\overline{M}}\,\overline{V}_{Qi0} \rangle.$$

4.2 Perturbation of capacitances

Under infinitesimal perturbations, we have (constant coefficients have been discarded for clarity):

$$d\overline{\overline{M}} \sim \begin{vmatrix} \Sigma_1^4\,dC_i & dC_1 - dC_3 & dC_4 - dC_2 \\ dC_1 - dC_3 & dC_1 + dC_3 & 0 \\ dC_4 - dC_2 & 0 & dC_2 + dC_4 \end{vmatrix}$$

and clearly the first column will act on Q modes. The equations: $\begin{cases} C_1 = C_{QQ} + C_{SQ} \\ C_2 = C_{QQ} - C_{TQ} \\ C_3 = C_{QQ} - C_{SQ} \\ C_4 = C_{QQ} + C_{TQ} \end{cases}$ define a projection into a subspace of vector dimension = 3, and let us write the three following formal expansions:

$$C_{XQ} =: \sum_m p_{XQm}\,\xi_{XQm}(z) \quad \text{with} \quad \xi_{XQm} = \frac{V_{Xm}}{(\partial M_{XQ}/\partial C_{XQ})V_{Qi0}}$$

and $X = Q, S$ or T.

Then: $\langle \overline{V}_{Xi}\,, d\overline{\overline{M}}\,\overline{V}_{qi0} \rangle = dp_{XQi}$, $X = Q, S$ or T.

The application:

$$\{\overline{V}_{Qi}\}, \{\overline{V}_{Sj}\}, \{\overline{V}_{Tk}\} \rightarrow \{\overline{\xi}_{Qi}\}, \{\overline{\xi}_{Sj}\}, \{\overline{\xi}_{Tk}\}$$

is a bounded linear bijection that maps the V functions to the ξ functions for every order. Thus [5] the subset of ξ functions is a Riesz basis for the Hilbert space defined with $H^2(\Omega)$ and the norm defined from the sesquilinear form: $(\overline{u}\,, \overline{v}) = \int_\Omega \overline{\overline{W}}^*\,\overline{u}^* \bullet \overline{v}\,d\Omega$, where the array $\overline{\overline{W}}$ is diagonal, with $W_{QQ} = [\,(\partial M_{SQ}/\partial C_{SQ})V_{Qi0}\,]^2$ etc. On a practical point of view, measured voltages may be expanded on the V basis, and transformed directly into capacitance perturbations. The boundary conditions are numerically extracted from the data, and used to compute the V basis: the perturbation theory applies to a perfectly symmetric line, with boundary conditions of the self-adjoint type, equivalent to the real, but unknown, conditions.

4.5 Inductance perturbations

The effect of inductance perturbations on the operator is similar to that of capacitances. In the same way we define the projection:

$\{L_1, L_2, L_3, L_4\} \to \{L_{QQ}, L_{SQ}, L_{TQ}\}$.

Now the basis functions take the form of rectangular pulses: $\xi_{XQm} = \Sigma_t\, \xi_{XQtm}\, u_t(z)$,

where t is the tuner index, m the mode index, $u_t(z)$ the characteristic function of tuner #t, and X = Q, S or T. Let:

$$K_{mt} = \int_\Omega V_{Xm}\, (\partial M_{XQ}/\partial L_{XQ})\, u_t\, V_{qi0}\, dx \ .$$

The set of tuning planes locations $\{z_t\}_{t=1\ldots T}$, where T is the number of tuners per RFQ quadrant, has to be:

(i) a set of interpolation in the subset spanned by the first T eigen-functions,

i.e. $\overline{\overline{K}}_{it} \cdot \overline{\overline{\xi}}_{tj} = \overline{\overline{1}}$, i = 1...T , j = 1...T , t = 1...T ,

(ii) a set of stable sampling over the whole eigen-space,

i.e. $\Sigma_t\, (K_{it}\, \xi_{tj})^2 \le k\ , \forall j \le T\ , \forall i > T$,

for some constant k independent of m and i.

Of course, the ξ_{tm} may be computed with condition (i) alone. Numerical experience shows that a highly irregular distribution of tuner planes may lead to unstable sampling. Unfortunately, no 'sampling theorem' has been established to date, giving the necessary and sufficient conditions for a sampling set to satisfy both conditions (i) and (ii) in our functional spaces (despite the 'one-quarter' Kadec theorem that applies to irregular sampling of Paley-Wiener functions – see [5] for a thorough discussion). An anti-aliasing least-square (LS) filter is used to generate the discrete inductance basis functions:

$$\sum_{t=1\ldots T} K_{it}\ \xi_{tj} = \delta_{ij}\ ,\ i = 1\ldots T'\ ,\ j = 1\ldots T'\ ,\ T' \le T\ ,$$

$$\sum_{t=1\ldots T} K_{it}\ \xi_{tj} \overset{LS}{=} 0\ ,\ i = T'+1\ldots T''\ ,\ j = 1\ldots T'\ ,\ T'' \ge T\ ,$$

where [1, T'] is the modal pass-band, and [T'+1, T''] the modal stop-band (now, the tuners control only T' modes). In block-array form, the two equations are (the 'a' dimension is T', the 'b' dimension, T-T', and the 'c' dimension, T''-T):

$$\begin{cases} \overline{\overline{K}}_{aa}\, \overline{\overline{\xi}}_{aa} + \overline{\overline{K}}_{ab}\, \overline{\overline{\xi}}_{ba} = \overline{\overline{1}} \\ \overline{\overline{K}}_{ca}\, \overline{\overline{\xi}}_{aa} + \overline{\overline{K}}_{cb}\, \overline{\overline{\xi}}_{ba} \overset{LS}{=} \overline{\overline{0}} \end{cases},$$

yielding:

$$\begin{cases} \overline{\overline{\Gamma}} = \overline{\overline{K}}_{bb} - \overline{\overline{K}}_{ba}\, \overline{\overline{K}}_{aa}^{\,-1}\, \overline{\overline{K}}_{ab} \\ \overline{\overline{\xi}}_{ba} = -(\overline{\overline{\Gamma}}^{\,t}\, \overline{\overline{\Gamma}})^{-1}\, \overline{\overline{\Gamma}}^{\,t}\, \overline{\overline{K}}_{ba}\, \overline{\overline{K}}_{aa}^{\,-1} \\ \overline{\overline{\xi}}_{aa} = \overline{\overline{K}}_{aa}^{\,-1}\, (\overline{\overline{1}} - \overline{\overline{K}}_{ab}\, \overline{\overline{\xi}}_{ba}) \end{cases}.$$

5 END AND COUPLING CELLS

5.1 End cells

The general form of an end cell boundary condition is:

$$\partial_z \overline{U} \equiv \begin{vmatrix} \partial_z U_Q \\ \partial_z U_S \\ \partial_z U_T \end{vmatrix} = \begin{vmatrix} s_{QQ} & s_{QS} & s_{QT} \\ s_{SQ} & s_{SS} & s_{ST} \\ s_{TQ} & s_{TS} & s_{TT} \end{vmatrix} \begin{vmatrix} U_Q \\ U_S \\ U_T \end{vmatrix} \equiv \overline{\overline{s}}\, \overline{U}$$

which depends on 9 parameters. One RFQ measurement yields one pair of values $\{\overline{U}, \partial_z\overline{U}\}$, and so a set of three equations. Three voltage-independent measurements would yield the 9 required equations. The 3 excitations are provided by moving the tuners in a coordinated way, at some distance of the end cell (the estimation of voltage slopes needs clean data over some distance). The singular value decomposition of the recovered array is:

$\overline{\overline{s}} = \overline{\overline{F}}\ \overline{\overline{s}}_d\ \overline{\overline{G}}^{\,t}$, where:

$\overline{\overline{F}}$, $\overline{\overline{G}}$ = orthonormal base transform arrays,

$\overline{\overline{s}}_d$ = diagonal array of (positive) singular values.

In the new axes: $\partial_z\overline{U}'' = \overline{\overline{s}}_d\ \overline{U}'$.

The tuning of the end cell is directly related to the amplitude of the singular values (s_{dQQ} for the desired quadrupole mode), and the relative orientation of the {Q'',S'',T''} axes with respect to the {Q',S',T'} axes illustrates the symmetry errors. More than 3 data sets may be used to derive mean singular values, in a least-square sense. The standard deviation is estimated from the collection of all possible triplets, and will be smaller if the voltage data sets are widely dispersed on the {Q,S,T} sphere.

5.2 Coupling cells

The same procedure is applied, in dimension 6:

$$\begin{vmatrix} \partial_z U_{\bar{Q}} \\ \partial_z U_{\ddagger Q} \\ \partial_z U_{\bar{S}} \\ \partial_z U_{\ddagger S} \\ \partial_z U_{\bar{T}} \\ \partial_z U_{\ddagger T} \end{vmatrix} = \overline{\overline{s}} \begin{vmatrix} U_{\bar{Q}} \\ U_{\ddagger Q} \\ U_{\bar{S}} \\ U_{\ddagger S} \\ U_{\bar{T}} \\ U_{\ddagger T} \end{vmatrix} .$$

For a perfectly symmetric coupling cell, the $\overline{\overline{s}}$ array is 3x3 block-diagonal, and a little more algebraic processing is needed to recover the block coefficients from the singular values diagonal array.

6 REFERENCES

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