CORRECTION TERMS TO PANOFSKY-WENZEL FORMULA AND WAKE POTENTIAL

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Abstract

In their 1956 article [1] Panofsky and Wenzel derived a relation for the net transverse kick experienced by a fast charge particle crossing a closed cavity excited in a single rf mode. Later this relation, usually referred to the Panofsky-Wenzel theorem, was generalized for cavity containing wake field induced by a driving charge. This theorem has played very important role in accelerator physics. One well-known conclusion of this paper was that in a TE mode the deflecting impulse of the electric field always cancels the impulse of the magnetic fields. In our presentation we more exactly rederive Panofsky and Wenzel’s relation and obtain correction terms to it. As well we will discuss possibility to measure phase volume of a bunch with rf deflector based on a TE mode. Finally we will attempt to find correction terms to the wake potential.

REDERIVING THE THEOREM

Following to the Panofsky-Wenzel derivation [1], the equation of motion of the particle in terms of a vector potential is given

\[
\frac{dp}{dz} = e \left[ \frac{\partial A}{\partial t} + \vec{v} \times \vec{\nabla} \times \vec{A} \right], \quad (2)
\]

where \( dz = v dt \). Using the following expressions

\[
\vec{v} \times \vec{\nabla} \times \vec{A} = \vec{\nabla} \left( \vec{v} \cdot \vec{A} \right) - \left( \vec{v} \vec{\nabla} \right) \vec{A}, \quad \frac{\partial A}{\partial t} = \frac{dA}{dt} - \left( \vec{v} \vec{\nabla} \right) \vec{A}, \quad \text{and}
\]

expressing the particle velocity as \( \vec{v} = \vec{v}_z + v_z \vec{p}_z / p_z \), (where \( \vec{p}_z \) and \( p_z \) are the transverse and longitudinal momentums, respectively) we can write the equation for transverse momentum as

\[
\frac{dp}{dz} = e \left[ -\frac{dA}{dz} + \vec{\nabla} \left( A_z + \frac{\vec{p}_z}{p_z} \vec{A} \right) \right]. \quad (3)
\]

Integrating Eq.(3) we obtain the dependence of the transverse momentum on a coordinate \( z \)

\[
\vec{p}_z(z) = \vec{p}_{z0} - e \vec{A}_z \left( \vec{r}_z, t, z \right) \Bigg|_{z=0} + e \int_{0}^{z} \vec{\nabla} \left( A_z + \frac{\vec{p}_z}{p_z} \vec{A} \right) \left( \vec{r}_z, t, z \right) dz, \quad (4)
\]

where it is assumed that \( \vec{A}_z = 0 \) at \( z=0 \) and \( z=L \) (the cavity end walls are normal the z-diraction or the path of the particle begins and ends in a field-free region), \( \vec{p}_{z0} \) is the initial transverse momentum, \( \vec{r}_z \) is the transverse coordinate of the charge. Due to the small parameter \( p_z / \vec{p}_z \sim 1 \), the integral equation Eq.(4) may be solved by the successive approximations. Therefore we expand it into series on the small parameter

\[
\vec{p}_z = \vec{p}_{z0} - e \left[ 1 + \delta \vec{r}_z \cdot \vec{\nabla} \right] \vec{A}_z \left( \vec{r}_{z0}, t, z \right) \Bigg|_{z=0} + e \int_{0}^{z} \left[ 1 + \delta \vec{r}_z \cdot \vec{\nabla} \right] A_z \left( \vec{r}_{z0}, t, z \right) + \frac{\vec{p}_z}{p_z} \vec{A} \left( \vec{r}_{z0}, t, z \right) \Bigg|_{z=0} dz \quad (5)
\]

In the case of \( A_z=0 \) or \( v_z A_z \parallel \vec{v}_z \vec{A}_z \), it is necessary to take into account the transverse momentum imparted to the particle during its transit time through the cavity. In this paper we will derive more exactly the Panofsky and Wenzel’s relation and obtain correction terms to it. As well we will discuss possibility to measure phase volume of a bunch with rf deflector based on a TE mode. Finally we will attempt to find correction terms to the wake potential.

INTRODUCTION

The well-known Panofsky-Wenzel formula [1] is concerned with the net transverse kick experienced by a fast charged particle crossing a closed cavity containing rf fields

\[
\Delta p_z = e \int_{0}^{L} \vec{\nabla} \cdot \vec{A} \Bigg|_{z=0} dz. \quad (1)
\]

It is the practical tool in dynamic of ultrarelativistic beams interacting with rf structures. In Eq.(1) \( e \) is the charge of particle, \( v_z \) is the longitudinal velocity close to the speed of light \( c \), \( L \) is the length of cavity, \( \vec{\nabla} \cdot \vec{A} \) is the transverse gradient of the longitudinal component of rf vector potential. In the wake potential theory the relation (1), usually referred to as the Panofsky-Wenzel theorem, was generalized for rf cavities and infinitely repeating structures containing wake field induced by a driving charge [2]. Some reformulated versions of the Panofsky-Wenzel theorem are given in Ref. [3] for study of rf asymmetry in photo-injectors, in Ref. [4] for the case in which phase slippage between the wave and beam is not negligible. The interesting interpretation of paper [1] results can be found in [5].

One well-known conclusion, that in a TE mode the deflecting impulse of the electric field always cancels the impulse of the magnetic fields, follows from Eq.(1). However, generally, if \( A_z \) is zero or small enough, the formula (1) is not true. The fact is that the Panofsky-Wenzel theorem assumes in its derivation that the particle experiencing Lorentz force moves parallel to the \( z \)-axis at constant velocity \( \vec{v} = v_z \vec{v}_z + \vec{v}_z \vec{A}_z \). So, in the dot product \( \vec{v} \vec{A} = v_z A_z + \vec{v}_z \vec{A}_z \) the second term was neglected.

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Here \( \delta r_z = \left[ (\bar{p}_z / p_z) dz \right] \) and it is assumed that

\[
\left[ \delta r_z, \delta \dot{r}_z \right] = [\bar{r}_z, \bar{r}_z] \text{ is the initial transverse coordinate of the charge. Firstly from Eq.(5) we find the zero order approximation of the transverse momentum as function of z-coordinate}
\]

\[\bar{p}_z = \left( \bar{p}_{0,z} - e \bar{A}_{L} \right) z, t \right]_{z = \gamma \nu}, \quad (6)
\]

We see that at \( z = L \) the zero order approximation Eq.(6) reduces to the Panofsky-Wenzel formula (1). Substituting Eq.(6) into Eq.(5) we obtain the transverse momentum

\[
\Delta \bar{p}_z = e \int_0^L \left( \frac{\bar{A}_z}{p_z} - \frac{|e| A_z}{p_x} \right) dz \nabla \cdot A_z, \quad (7)
\]

\[
\int_0^L \left( \frac{\bar{p}_{0,z} e \bar{A}_z}{p_z} - \frac{|e| A_z}{p_x} + e \left( \bar{A}_z \right)^2 \right) \nabla \cdot A_zdz = 0
\]

\[
\nabla \cdot A_z = \psi \frac{e}{m} \nabla \cdot A_z,
\]

where \( \bar{A} = (\bar{p}_{0,z}, \bar{z}, t) \). From Eq.(7) we see that in the case of exciting a TE mode \( \bar{A} = 0 \) the net transverse kick is

\[
\Delta \bar{p}_z = \int_0^L \left( \frac{\bar{p}_{0,z} e \bar{A}_z}{p_z} - \frac{|e| A_z}{p_x} \right) \nabla \cdot A_zdz \quad (8)
\]

As seen from the Eq.(8), even if \( \bar{p}_{0,z} = 0 \), the ponderomotive force, which is square on the transverse component of vector potential, ensures the non-zero transverse momentum imparted to the particle.

**CONCEPT OF MEASUREMENT OF PHASE VOLUME BY TE MODE DEFLECTOR**

Eq.(8) shows that the transverse momentum imparted to the particle by a TE mode dependsences on the initial transverse momentum. That may point to ability to measure phase volume of a beam by using a TE mode deflector. Let us rewrite Eq.(8) in components as

\[
v_{0,x} a_{xx} + v_{0,y} a_{xy} = f_x, \quad v_{0,x} a_{yx} + v_{0,y} a_{yy} = f_y, \quad (9)
\]

where

\[
\begin{align*}
\bar{a}_{xx} &= e \int_0^L \frac{\partial A_z}{\partial x} dz, & a_{yy} &= e \int_0^L \frac{\partial A_z}{\partial y} dz, \\
\bar{a}_{yx} &= e \int_0^L \frac{\partial A_z}{\partial y} dz, & a_{xy} &= e \int_0^L \frac{\partial A_z}{\partial x} dz
\end{align*}
\]

\[
f_x = e^2 \int_0^L \frac{\partial A_z^2}{\partial x} dz - \Delta p_x, \quad (10)
\]

\[
f_y = e^2 \int_0^L \frac{\partial A_z^2}{\partial y} dz - \Delta p_y, \quad (11)
\]

The solution of the equation set (9) is

\[
v_{0,x} = \frac{f_x}{a_{xx}}, \quad v_{0,y} = \frac{f_y}{a_{yy}}. \quad (12)
\]

**An ultrarelativistic beam**

For a case of ultrarelativistic particles, \( v_z, \gamma \to \infty \), where \( \gamma \) is Lorentz factor ) Eqs.(12) can be simplified

\[
\frac{\Delta x}{\Delta t} = \frac{m \nu^2 \gamma}{c} \quad (13)
\]

Here the transverse kick \( \Delta p_x, \Delta p_y \) is expressed through a beam deflection from axis \( \Delta x = x - x_0, \Delta y = y - y_0 \) in a drift tube of length \( l \) which is stationed after the cavity, \( \Delta p_x = m \nu \gamma \Delta x / \gamma / \Delta y, \Delta p_y = m \nu \gamma \Delta y / \gamma / \Delta x \), \( m_0 \) is the rest mass, \( (x_0, y_0) \) are the transverse coordinates of a particle at the entry of the cavity and the drift tube exit, correspondently.

Let us consider a rectangular box where the following TE modes are excited

\[
A_{n,m} = B_0 \left( \frac{\pi x}{a} \right), \quad \text{with the eigenfrequency } \omega_{n,m} = \frac{\pi c \sqrt{m a^2 + b^2}}{a}, \quad (15)
\]

\[
A_{n,m} = B_0 \left( \frac{\pi y}{b} \right), \quad \text{with the eigenfrequency } \omega_{n,m} = \frac{\pi c \sqrt{m a^2 + b^2}}{b}, \quad (15)
\]

where \( a, b, L \) are the edge lengths of the rectangular box in \( x, y \), and \( z \)-directions, correspondently. \( B_0 \) is the constant, \( \phi \) is the initial phase.

We write

\[
x = \langle x \rangle + \delta x, \quad y = \langle y \rangle + \delta y, \quad \delta x = \delta \langle x \rangle, \quad \delta y = \delta \langle y \rangle
\]

where \( \langle \ldots \rangle \) is the operator of averaging over particles, \( \langle \phi \rangle \) is the reference phase.

We assume that the bunch to be short \( |\delta \phi| < 2\pi \). Setting \( \langle x_0 \rangle = a / 2, \langle y_0 \rangle = b / 2 \), and substituting Eq.(14, 15) for even \( n \) and \( m \) into Eq. (13) we obtain

\[
\begin{align*}
\frac{\Delta y}{\Delta t} = \gamma \bar{\beta}_{n,m} \left( \frac{e B_0}{m c L} \right)^2 \frac{(p_x / L)^2}{n} \cos \phi \left[ (1 - \sin \phi \left( \frac{\omega_{n,m} L}{c} + \phi \right) \right], \\
\frac{\Delta x}{\Delta t} = \gamma \bar{\beta}_{n,m} \left( \frac{e B_0}{m c L} \right)^2 \frac{(p_y / L)^2}{n} \cos \phi \left[ (1 - \sin \phi \left( \frac{\omega_{n,m} L}{c} + \phi \right) \right],
\end{align*}
\]

(17)
\[ \gamma \frac{\Delta \nu}{\tau} = \beta_{0,\nu} \left( -1 \right)^{\nu} \frac{eB_0}{m_c \gamma} \left( \frac{\alpha \omega_{L} L}{c^2} \phi \right) \left[ \cos \phi_{c} - \left( -1 \right)^{\nu} \cos \left( \frac{\alpha \omega_{L} L}{c^2} \phi \right) \right]. \]

Setting the reference phase \( \phi_{r,k} \) (where \( k=n,m \)) at which
\[ \sin \phi_{r,k} - \left( -1 \right)^{\nu} \sin \left( \frac{\alpha \omega_{L} L}{c^2} \phi_{r,k} \right) = 0, \]
we can obtain the initial transverse characteristic bunch
\[ \langle \beta_{0,\nu} \rangle = \langle y^2 \rangle - \langle y^2 \rangle - 2 \langle y_0 \rangle (\langle y \rangle - \langle y_0 \rangle). \]

Using the approach developed above we consider wake fields \( (\hat{E}, \hat{B}) \) in terms of vector and scalar potentials \( \hat{A}, \Phi \)
\[ \hat{E} = -\frac{\partial \hat{A}}{\partial t} + \nabla \Phi, \quad \hat{B} = \nabla \times \hat{A}, \quad \nabla \hat{A} = 0. \]

\[ d\hat{p} = e \left\{ -\frac{d\hat{A}}{dz} + \frac{1}{\gamma c} \left[ \hat{A}_{t} - \frac{\Phi}{\rho_{t}} + \frac{\rho_{t}}{\gamma} \right] \right\} dz. \] (25)

Integrating Eq.(25) over \( (0, z) \), then, expanding \( \hat{p} \) into series on the small parameter \( p_{z}/p_{t} \ll 1 \), we find the zero order transverse momentum as function of \( z \)
\[ \hat{p}^{(0)}(z,s) = \hat{p}_{0,\perp} - e\hat{A}_{\perp}(z,s) + e \int_{0}^{s} \nabla_{\perp} \left( \frac{\hat{A}_{t}}{\rho_{t}} - \frac{\Phi}{\rho_{t}} \right) dz, \] (26)

Further for simplicity we assume that the path of the particle begins and ends in a field-free region, \( \hat{A} = 0 \). Substituting Eq.(26) into Eq.(25), and taking into account the definition [2], we obtain the wake potential with the correction terms
\[ \tilde{W}(s) = \frac{V A_{\perp}(s)}{eq} + \frac{1}{q} \int_{0}^{s} \left( \nabla_{\perp} (v_{t} A_{t} - \Phi) + u \right) dz, \] (27)

where \( u \) associates the correction terms
\[ u = \int_{0}^{s} \rho_{0,\perp} \cdot \nabla_{\perp} \left( v_{t} A_{t} - \Phi \right) dz, \] (28)

Substituting Eq.(26) into Eq.(28) we obtain
\[ u = \int_{0}^{s} \rho_{0,\perp} \cdot \nabla_{\perp} \left( v_{t} A_{t} - \Phi \right) dz. \] (29)

As seen from the Eq.(29) the correction terms to the wake potential are proportional to \( \gamma^{-1} \), whereas the modern wake theory [2] gives the correction terms which are proportional to \( \gamma^{-2} \).

**SUMMARY**

Rederiving the Panofsky and Wenzel’s theorem we obtained the correction terms to the net transverse kick which is not zero in the TE mode. That allows to use the TE mode deflector to measure phase volume of a beam. We found the correction terms to wake potentials which are shown to be inversely proportional to the relativistic factor.

**REFERENCES**