AMPLIFICATION OF CURRENT DENSITY MODULATION IN A FEL WITH AN INFINITE ELECTRON BEAM*

G. Wang#, V. N. Litvinenko, S. D. Webb, BNL, Upton, NY 11973, U.S.A.

Abstract
We show that the paraxial field equation for a free electron laser (FEL) in an infinitely wide electron beam with \( \kappa - 2 \) energy distribution can be reduced to a fourth ordinary differential equation (ODE). Its solution for arbitrary initial phase space density modulation has been derived in the wave-vector domain. For initial current modulation with Gaussian profile, close form solutions are obtained in space-time domain.

INTRODUCTION

In developing an analytical model for a FEL-based coherent electron cooling system, an infinite electron beam has been assumed for the modulation and correction processes [1-3]. While the assumption has its limitation, it allows for an analytical close form solution to be obtained, which is essential for investigating the underlying scaling law, benchmarking the simulation codes and understanding the fundamental physics.

1D theory was previously applied to model a CeC FEL amplifier[4]. However, the theory ignores diffraction effects and does not provide the transverse profile of the amplified electron density modulation. On the other hand, 3D theories developed for a finite electron beam usually have solutions expanded over infinite number of modes determined by the specific transverse boundary conditions. Unless the mode with the largest growth rate substantially dominates other modes, both evaluation and extracting scaling laws can be complicated. Furthermore, it is also preferable to have an analytical FEL model with assumptions consistent with the other two sections of a CeC system.

Recently, we developed the FEL theory in an infinitely wide electron beam with \( \kappa - 1 \) (Lorentzian) energy distribution[5]. Close form solutions have been obtained for the amplified current modulation initiated by an external electric field with various spatial-profiles. In this work, we extend the theory into \( \kappa - 2 \) energy distribution and study the evolution of current density induced by an initial density modulation.

EQUATION OF MOTION

Assuming that the amplitude of the radiation field varies slowly with respect to the undulator period and that fast oscillation terms can be dropped, the amplitude of the radiation field is determined by the following paraxial field equation [6]

\[
\left( V_z^2 + 2i \frac{\alpha}{c} \frac{\partial}{\partial z} \right) \tilde{E}(\rho,z) = i j_0(z) \int_0^{\infty} \left[ \frac{e^{imz}}{2 \epsilon_0} \tilde{E}(\rho_0, z_0) + \frac{e}{\epsilon_0 \omega} \left( V_z^2 + 2i \frac{\alpha}{c} \frac{\partial}{\partial z} \right) \tilde{E}(\rho_0, z_0) \right] \frac{e^{i \frac{\omega}{c} z_0 \tau}}{\sqrt{\pi \sigma_z \sigma_{\tau}}} d\rho_0 \]  

where \( \tilde{E}(\rho,z) \) is the complex amplitude of the radiation field, \( \omega \) is the radiation frequency, \( C \) is the electron energy deviation, \( E_0 \) is the energy distribution function, \( j_0(\rho) \) is the transverse spatial distribution of the unperturbed electron beam and \( j_0(\rho,0) \) is the initial phase space density perturbation. Assuming \( j_0(\rho) = j_0 \) is independent of \( \rho \) and performing Fourier transformation to eq. (1) lead to [5]

\[
\frac{\partial}{\partial z} \tilde{R}(z', k, \lambda) = i d^2 e^{i(\lambda - i)} \left[ \tilde{R}(z', k, \lambda) + i N_c \frac{\partial}{\partial z} \tilde{R}(z', k, \lambda) \right] d\lambda e^{i(\lambda - i)} + \frac{e \theta e}{2 \epsilon_0 \lambda} \frac{\partial}{\partial \lambda} \tilde{R}(z', k, \lambda) e^{i(\lambda - i)} d\lambda
\]

where

\[
\tilde{R}(z, k, \lambda) = e^{i \frac{\lambda}{\gamma_c \sigma_{\lambda}}} \tilde{E}(z, k, \lambda).
\]

In eq. (2), we used the normalized variables as defined in [6] and [5], i.e. \( z = z \gamma_c, \lambda = C/\Gamma, \lambda = \gamma_c \), the 1D- gain parameter

\[
\Gamma = \left[ \frac{\pi \xi_i \theta e \omega}{\gamma_c^2 \epsilon A_d} \right]^{1/2}
\]

the Alfven current \( I_A = m_c \gamma_c / e \), the Pierce parameter \( \rho = \gamma_i / \Gamma c / \omega \), the space charge parameter

\[
\lambda_p = \frac{1}{\Gamma^2} \frac{4 \pi \xi_i}{\gamma_i \epsilon A_d}
\]

the normalized transverse wave vector

\[
\tilde{k}_\perp = \frac{\hat{p} \hat{k}_\perp}{2 \gamma_i \Gamma},
\]

and the normalized energy distribution function \( \tilde{F}(\hat{p}) \) satisfying

\[
\int_{-\infty}^{\infty} \tilde{F}(\hat{p}) d\hat{p} = 1.
\]

We assume the following \( \kappa - 2 \) energy distribution,

\[
\rho(\omega) = \frac{1}{\sqrt{\pi \sigma_\epsilon \sigma_\lambda}} e^{-\frac{\omega^2}{\sigma_\epsilon^2}} e^{-\frac{\lambda^2}{\sigma_\lambda^2}} d\omega.
\]

* Work supported by Brookhaven Science Associates, LLC under Contract No.DE-AC02-98CH10886 with the U.S. Department of Energy.
# gawang@bnl.gov

Light Sources and FELs
Accel/Storage Rings 06: Free Electron Lasers

Proceedings of 2011 Particle Accelerator Conference, New York, NY, USA

THP149

Copyright © 2011 by PAC/IIUCIEE—cc Creative Commons Attribution 3.0 (CC BY 3.0)
Applying the relation
\[ \int d\hat{P} e^{i\hat{q} \hat{P}} \frac{\partial}{\partial \hat{P}} F(\hat{P}) = -i(\hat{q}^2 - 1) \left[ 1 + \hat{q} (1 - i) e^{i(\hat{q} \hat{P})} \right] \]
\[ \cdot \left[ 1 + \hat{q} (1 - i) e^{-i(\hat{q} \hat{P})} \right] \]
\[ \cdot \int_{-\infty}^{\infty} \mathcal{J}_i(\hat{P} \cdot \hat{\mathcal{P}}) d\hat{P} \]
\[ = -\hat{q} \int d\hat{z} e^{i\hat{q} \hat{z} \hat{z}} \left[ \hat{\mathcal{C}}_{\lambda} + 3i \mathcal{R}(\hat{z}, \hat{k}, \hat{C}) \right] \]
\[ \cdot \left[ \hat{\mathcal{C}}_{\lambda} + 3i \mathcal{R}(\hat{z}, \hat{k}, \hat{C}) \right] \]
\[ \cdot \left[ 1 + \hat{q} (1 - i) e^{i(\hat{q} \hat{P})} \right] \]
\[ \cdot \left[ 1 + \hat{q} (1 - i) e^{-i(\hat{q} \hat{P})} \right] \]
\[ \cdot \int_{-\infty}^{\infty} \mathcal{J}_i(\hat{P} \cdot \hat{\mathcal{P}}) d\hat{P} \]

where we define
\[ \hat{\mathcal{C}}_{\lambda} = \hat{\mathcal{C}} - \hat{\mathcal{C}}_{\lambda} \]
(10)

The third derivative of eq. (9) with respect to \( \hat{z} \) reads
\[ \mathcal{R}^{(4)} + 3i \left( \mathcal{C}_{\lambda,\lambda} + \hat{q} \right) \mathcal{R}^{(3)} + \left[ \hat{\mathcal{C}}_{\lambda} + 3i \mathcal{R}(\hat{z}, \hat{k}, \hat{C}) \right] \mathcal{R}^{(2)} \]
\[ + \left[ i \mathcal{C}_{\lambda,\lambda} + 3i \hat{q} \right] \mathcal{R}^{(1)} \]
\[ - i \left( \mathcal{C}_{\lambda,\lambda} + 3i \hat{q} \right) \mathcal{R}^{(0)} = \frac{1}{n_0} \int_{-\infty}^{\infty} \left( \hat{q} - i \hat{P} \right) \mathcal{J}_i(\hat{P}, 0) e^{-i(\hat{q} \hat{P})} d\hat{P} \]
\[ + \left[ i \mathcal{C}_{\lambda,\lambda} + 3i \hat{q} \right] \mathcal{R}^{(0)} \]
\[ - i \left( \mathcal{C}_{\lambda,\lambda} + 3i \hat{q} \right) \mathcal{R}^{(0)} = \frac{1}{n_0} \int_{-\infty}^{\infty} \left( \hat{q} - i \hat{P} \right) \mathcal{J}_i(\hat{P}, 0) e^{-i(\hat{q} \hat{P})} d\hat{P} \]
\[ + \left[ i \mathcal{C}_{\lambda,\lambda} + 3i \hat{q} \right] \mathcal{R}^{(0)} \]
\[ - i \left( \mathcal{C}_{\lambda,\lambda} + 3i \hat{q} \right) \mathcal{R}^{(0)} = \frac{1}{n_0} \int_{-\infty}^{\infty} \left( \hat{q} - i \hat{P} \right) \mathcal{J}_i(\hat{P}, 0) e^{-i(\hat{q} \hat{P})} d\hat{P} \]

and the summation is over the cyclic permutation of the four indices. The particular solution of eq. (11) is
\[ \mathcal{R}_p(\hat{z}) = \int_{0}^{\hat{z}} G(\hat{z} - \hat{z}_0) f(\hat{z}_0) d\hat{z}_0 \]
\[ = \frac{e^{i\hat{q} \hat{z}}}{2\pi \epsilon_0 \left( \lambda_1 - \lambda_2 \right) \left( \lambda_3 - \lambda_4 \right) \left( \lambda_5 - \lambda_6 \right)} \]
\[ \times \left[ \hat{q} - i \hat{P} \right] \left[ e^{i\hat{q} \hat{z}} - e^{-i(\hat{q} \hat{P})} \right] \mathcal{J}_i(\hat{P}, 0) d\hat{P} \]
(14)

Since eq. (14) satisfies
\[ \mathcal{R}(0) = \mathcal{R}^{(0)}(0) = \mathcal{R}^{(1)}(0) = \mathcal{R}(1)(0) = 0 \]
\[ \mathcal{R}_p(\hat{z}) = \sum_{\lambda, i} \lambda_i A_i \hat{\mathcal{C}}_{\lambda} \hat{z}^{\lambda_i} \]
(15)

From eq. (3), the solution for radiation field is
\[ \mathcal{E}(z, \hat{k}, C) = e^{-i\hat{q} \hat{z}} \left[ \mathcal{R}(z, \hat{k}, C) + \mathcal{R}(z, \hat{k}, C) \right] \]
(17)

The current density is related to the radiation field by [5, 6]
\[ \mathcal{J}_i(z, \hat{C}, \hat{k}) = -\frac{e\epsilon_0}{2\pi \epsilon_0} \left[ \hat{q}^2 + \frac{\partial}{\partial \hat{z}} \right] \mathcal{E} \]
(18)

Incorporating eq. (14), eq. (16), eq. (17) into eq. (18) leads to
\[ \mathcal{J}_i(z, \hat{C}, \hat{k}) = -\frac{2e\epsilon_0 \Gamma}{\theta_i} \sum_{\lambda} A_i \lambda_i \hat{\mathcal{C}}_{\lambda} \]
\[ \cdot \left[ \hat{q} + i \left( \hat{P} + \hat{\mathcal{C}}_{\lambda} \right) e^{-i(\hat{q} \hat{P})} \right] \mathcal{J}_i(\hat{P}, 0) d\hat{P} \]

where
\[ \left[ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \right] = -\frac{\theta_i}{2e \epsilon_0 \Gamma} M^{-1} \left[ \begin{array}{c} -2e \epsilon_0 \Gamma \mathcal{E} \\ \theta_i \\ \theta_i \end{array} \right] \]
(19)

with
\[ M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1 (\lambda_1 - i \hat{k}^2) & \lambda_2 (\lambda_2 - i \hat{k}^2) & \lambda_3 (\lambda_3 - i \hat{k}^2) & \lambda_4 (\lambda_4 - i \hat{k}^2) \\ \lambda_1 (\lambda_1 - i \hat{k}^2) & \lambda_2 (\lambda_2 - i \hat{k}^2) & \lambda_3 (\lambda_3 - i \hat{k}^2) & \lambda_4 (\lambda_4 - i \hat{k}^2) \\ \lambda_1 (\lambda_1 - i \hat{k}^2) & \lambda_2 (\lambda_2 - i \hat{k}^2) & \lambda_3 (\lambda_3 - i \hat{k}^2) & \lambda_4 (\lambda_4 - i \hat{k}^2) \\ \lambda_1 (\lambda_1 - i \hat{k}^2) & \lambda_2 (\lambda_2 - i \hat{k}^2) & \lambda_3 (\lambda_3 - i \hat{k}^2) & \lambda_4 (\lambda_4 - i \hat{k}^2) \end{pmatrix} \]
(21)

The linearized Vlasov equation for electrons in a straight field-free section reads [6]
\[ \frac{\partial}{\partial \hat{z}} \mathcal{J}_i + i(\hat{C} + \hat{\mathcal{P}}) \mathcal{J}_i = 0 \]
(22)

Assuming the electron beam goes through a field-free straight section before entering the undulator, the initial current modulation and its derivatives can be derived from eq. (22) as
\[ \frac{\partial}{\partial \hat{z}} (\hat{z}) \mathcal{J}_i(\hat{z}, \hat{P}) = \frac{1}{\epsilon_0} \mathcal{J}_i(0, \hat{C}, \hat{k}, \hat{P}) \]
(23)

In order to proceed further, the initial phase space density modulation has to be specified.

**GAUSSIAN PROFILE**

For simplicity, we ignore the energy modulation and assume the initial phase space density perturbation has the following form
\[ \mathcal{J}_i(0, \hat{C}, \hat{k}, \hat{P}) = \frac{1}{\epsilon_0} \mathcal{J}_i(0, \hat{C}, \hat{k}, \hat{P}) \]
(24)

Inserting eq. (20), eq. (21), eq. (23) and eq. (24) into eq. (19) produces
\[ \mathcal{J}_i(z, \hat{C}, \hat{k}) = -\frac{e^{-i\hat{q} \hat{z}}}{\epsilon_0} \sum_{\lambda, \lambda_i} \lambda_i \left[ \lambda_1 - \lambda_2 \right] \left[ \lambda_3 - \lambda_4 \right] \left[ \lambda_5 - \lambda_6 \right] \]
\[ \times \left[ \lambda_1 (\lambda_1 - i \hat{k}^2) + i \hat{q} \hat{C}_{\lambda} \right] e^{-i(\hat{q} \hat{P})} \]
\[ \left( \lambda_1 + i \hat{k} \hat{C}_{\lambda} \right) e^{-i(\hat{q} \hat{P})} \]
(25)
Assuming the initial current density perturbation has Gaussian spatial profile, i.e.,

\[ j_0(z, t, x, y) = \frac{e_G}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right) \exp\left(-\frac{x^2 + y^2}{2\sigma_r^2}\right) \]

for \( z \leq 0 \), the Fourier components of (27) at \( z = 0 \) is

\[ \tilde{j}_1(0, k_x, k_y, \tilde{C}) = e^{\frac{i \tilde{C}^2}{\beta}} \cdot e^{-\frac{Q^2}{2\sigma_r^2}} \approx e \cdot e^{-\frac{Q^2}{2\sigma_r^2}} \cdot e^{-\frac{Q^2}{2\sigma_r^2}} \]

Inserting eq. (28) into eq. (25) and carrying out the inverse Fourier transformations yields[4]