**Periodically-Focused Solutions to the Nonlinear Vlasov-Maxwell Equations for Intense Beam Propagation Through an Alternating-Gradient Quadrupole Field**

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Abstract

This paper considers an intense nonneutral ion beam propagating in the $z$-direction through a periodic focusing quadrupole field with transverse focusing force, $\mathbf{F}_{\text{foc}} = -\kappa_q(s)(x \mathbf{e}_x - y \mathbf{e}_y)$, on the beam ions. A third-order Hamiltonian averaging technique using a canonical transformation is employed to transform away the rapidly oscillating terms. This leads to a Hamiltonian, $\mathcal{H}(X, Y, X', Y', s) = (1/2)(X'^2 + Y'^2) + (1/2)\kappa_f(q(X^2 + Y^2) + \psi(X, Y, s)$, in the transformed variables $(X, Y, X', Y')$, where the focusing coefficient $\kappa_f$ is constant, and many solutions and properties of the Vlasov-Maxwell system are well known.

1 INTRODUCTION

It is important to be able to investigate, based on the nonlinear Vlasov-Maxwell equations, the equilibrium and stability properties of general distribution functions for periodically-focused beams[1, 2, 3]. Despite its limited practical interest due to the unphysical distribution in phase space, the Kapchinskij-Vladimirskij (KV) beam equilibrium[1, 4, 5, 6], including its recent generalization to a rotating beam in a periodic focusing solenoidal field[7, 8], has been the only known periodically-focused equilibrium solution to the nonlinear Vlasov-Maxwell equations describing an intense beam propagating through a periodic focusing field. The difficulty of solving the nonlinear Vlasov-Maxwell system in general lies in the fact that the Hamiltonian for the motion of an individual beam particle is time dependent. Channell[9] and Davidson et al[10] have recently developed a third-order Hamiltonian averaging technique using a canonical transformation to average over the fast time scale associated with the betatron oscillations.

2 CANONICAL TRANSFORMATION

Because of the oscillatory time dependence of $\kappa_q(s)$, there is no general analytical method to solve the nonlinear Vlasov-Maxwell equations. However, we can average over the fast time scale associated with the betatron oscillations when the phase advance is sufficiently small. The averaging process is accomplished by introducing a canonical coordinate transformation from the laboratory coordinate system $(x, y, x', y')$ to a new coordinate system $(X, Y, X', Y')$. In the laboratory coordinates, the single-particle Hamiltonian $H(x, y, x', y', s)$ is

$$H = \epsilon \left[ \frac{1}{2}(x'^2 + y'^2) + \frac{1}{2}\kappa_q(s)(x^2 - y^2) + \psi(x, y, s) \right],$$

where $\epsilon$ is a small dimensionless parameter proportional to the focusing field strength. We use a near-identity canonical transformation $T : (x, y, x', y') \rightarrow (X, Y, X', Y')$ that is generated by a generating function of the Von Zeipel form, i.e.,

$$S(x, y, X', Y', s) = xX' + yY' + \sum_{n=1}^{\infty} \epsilon^n S_n(x, y, X', Y', s).$$
Consequently, the transformed Hamiltonian in the new variables $\mathcal{H}(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}', s)$ is given by

$$
\mathcal{H} = \sum_{n=1}^{\infty} \epsilon^n \mathcal{H}_n = H + \frac{\partial}{\partial s} S(x, y, X', Y', s) .
$$

(6)

The corresponding coordinate transformation is given by

$$
X = \frac{\partial S}{\partial X'} = x + \sum_{n=1}^{\infty} \epsilon^n \frac{\partial}{\partial X'} S_n(x, y, X, Y', s) ,
$$

(7)

$$
x' = \frac{\partial S}{\partial x} = X' + \sum_{n=1}^{\infty} \epsilon^n \frac{\partial}{\partial x} S_n(x, y, X', Y', s).
$$

The equations for $Y$ and $y'$ are similar in form. We choose, order by order, the generating function $S_n$, in such a way that $S_n$ is independent of the fast time scale associated with oscillations in $\kappa_q(s)$, and solve for the coordinate transformation iteratively when $S_0$ is known. Following the detailed algebra presented in Ref. [10], we obtain the transformed Hamiltonian correct to order $\epsilon^3$.

$$
\mathcal{H} = \frac{1}{2}(\tilde{X}^2 + \tilde{Y}^2) + \frac{1}{2}\kappa_{f_q}(\tilde{X}^2 + \tilde{Y}^2) + \psi(\tilde{X}, \tilde{Y}, s) ,
$$

(8)

where we have set $\epsilon = 1$. Here, $\kappa_{f_q}$ is defined in Eq. (11), and we have introduced the additional (canonical) fiber transformation to shifted velocity coordinates defined by

$$
\tilde{X} = X , \quad \tilde{X}' = X' - \langle \alpha_q \rangle X ,
$$

$$
\tilde{Y} = Y , \quad \tilde{Y}' = Y' + \langle \alpha_q \rangle Y .
$$

(9)

Similarly, correct to order $\epsilon^3$, we calculate the inverse coordinate transformation, $x = \tilde{X} + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3$, $x' = \tilde{X}' + \epsilon x_1' + \epsilon^2 x_2' + \epsilon^3 x_3'$, etc. Setting $\epsilon = 1$, this gives[10]

$$
x(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}', s) = [1 - \beta_q(s)] \tilde{X} + 2 \left( \int_{0}^{s} \frac{\partial \beta_q(s)}{\partial s} \right) \tilde{X}' ,
$$

$$
x'(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}', s) = [1 + \beta_q(s)] \tilde{X}' + \left( -\alpha_q(s) + \langle \alpha_q \rangle \right) + \langle \alpha_q \rangle \beta_q(s) - \alpha_q(s) \beta_q(s) - \left( \int_{0}^{s} \frac{\partial \delta_q(s)}{\partial s} \right) \tilde{X}'
$$

$$
+ \left( \int_{0}^{s} \frac{\partial \beta_q(s)}{\partial s} \right) \frac{\partial}{\partial \tilde{X}} \left( \tilde{X} \frac{\partial \psi(\tilde{X}, \tilde{Y})}{\partial \tilde{X}} - \tilde{Y} \frac{\partial \psi(\tilde{X}, \tilde{Y})}{\partial \tilde{Y}} \right) .
$$

(10)

The coordinate transformation can be easily obtained by solving Eq. (10) for $\tilde{X}$ and $\tilde{X}'$ in terms of $x$ and $x'$. The expressions for $y$ and $y'$ are identical in form to Eq. (10) provided we make the replacements $(x, x') \rightarrow (y, y')$ and $(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}') \rightarrow (Y, X, Y', X')$ and reverse the signs of $\alpha_q(s)$ and $\beta_q(s)$. In the above equations, $\alpha_q(s)$, $\beta_q(s)$, and $\delta_q(s)$ are defined in terms of the lattice function $k_q(s)$, which is assumed to have zero average, $\int_{0}^{s} ds k_q(s) = 0$, and odd half-period symmetry with $\kappa_q(s - S/2) = -\kappa_q(-s - S/2)$. The definitions are given by

$$
\alpha_q(s) = \int_{0}^{s} ds k_q(s) , \quad \beta_q(s) = \frac{1}{2} \int_{0}^{s} ds \left[ \alpha_q(s) - \langle \alpha_q \rangle \right] ,
$$

$$
\langle \alpha_q \rangle = \frac{1}{S} \int_{0}^{S} ds \langle \alpha_q \rangle .
$$

(11)

In addition, $\kappa_q(s)$ and $\langle \alpha_q \rangle$ are of order $c$; $\beta_q(s)$ is of order $c^2$; and $\langle \alpha_q \rangle \beta_q(s)$, $\alpha_q(s) \beta_q(s) \left( \int_{0}^{s} ds \beta_q(s) \right)$, and $\langle \alpha_q \rangle \beta_q(s)$ are of order $c^3$.

### 3 Vlasov-Maxwell Equations in the Transformed Variables

Because the transformation leading to the new Hamiltonian in Eq. (8) is canonical, the nonlinear Vlasov-Maxwell equations for the distribution function $F_b(x, \tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}')$ and self-field potential $\psi(\tilde{X}, \tilde{Y}, s)$ in the transformed variables are given by

$$
\left\{ \frac{\partial}{\partial s} + \tilde{X}' \frac{\partial}{\partial \tilde{X}} + \tilde{Y}' \frac{\partial}{\partial \tilde{Y}} - \left( \kappa_{f_q} \tilde{X} + \frac{\partial \psi(\tilde{X}, \tilde{Y})}{\partial \tilde{X}} \right) \frac{\partial}{\partial \tilde{X}'} \right\} F_b = 0 ,
$$

and

$$
\left( \frac{\partial^2}{\partial \tilde{X}^2} + \frac{\partial^2}{\partial \tilde{Y}^2} \right) \psi = -\frac{2\pi K_0}{N_b} \int d\tilde{X} ' d\tilde{Y} ' F_b ,
$$

(12)

(13)

where $\kappa_{f_q} = \text{const}$ is defined in Eq. (11). Variables in laboratory-frame coordinates can be obtained through the pull-back transformation $\tilde{T}^*$ associated with the coordinate transformation

$$
\tilde{T} : (x, y, x', y') \mapsto (\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}') .
$$

(14)

Here, $\tilde{T}^*$ pulls (transforms) functions on $(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}')$ back into functions on $(x, y, x', y')$. For example, the distribution function transforms according to

$$
\tilde{T}^* : F_b(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}', s) \mapsto f_b(x, y, x', y', s) .
$$

(15)

In addition, we obtain the following pull-back equation for the beam density correct to order $c^3$.

$$
n_b(x, y, s) = \int d\tilde{x} d\tilde{y} dx dy \quad (\tilde{x} = x + \sum_{n=1}^{N} \epsilon^n \frac{\partial}{\partial x} S_n(x, y, X', Y', s) ,
$$

$$
= \int d\tilde{x} d\tilde{y} d\tilde{x}' d\tilde{y}' F_b(\tilde{T}^{-1}\tilde{X} - x) \delta(\tilde{Y} - y)
$$

$$
= \left\{ \int d\tilde{x}' d\tilde{y}' [1 - (x_2 + x_3)] \frac{\partial}{\partial X} \right\} f_b(\tilde{x}, \tilde{y}, \rightarrow (x, y) .
$$

(16)
Here, \( x_2, y_2 \) and \( x_3, y_3 \), defined by Eq. (10), are the second-order and third-order inverse coordinate transformations expressed as functions of \((\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}')\).

Because of the simple form of the Vlasov-Maxwell equations in the transformed variables, with constant focusing coefficient \( \kappa_{f_q} = \text{const.} \), a wide range of literature developed for the constant focusing case\([1, 11, 12, 13]\) can be applied virtually intact in the transformed variables. For example, it is readily shown that any distribution function developed for the constant focusing case\([1, 11, 12, 13]\) can be applied virtually intact in the transformed variables. For example, to the leading order, the density profile is of the form\([10]\)

\[
\rho_b(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}') = F_b^0(\mathcal{H}^0),
\]

where \( \mathcal{H}^0 = (1/2)(\tilde{X}'^2 + \tilde{Y}'^2) + (1/2)\kappa_{f_q}(\tilde{X}^2 + \tilde{Y}^2) + \psi^0(\tilde{X}, \tilde{Y}) \) is the single-particle Hamiltonian, is an exact equilibrium solution to the Vlasov-Maxwell equations (12) and (13) with \( \partial / \partial s = 0 \). There is clearly enormous latitude\([1, 7]\) in specifying the functional form of \( F_b^0(\mathcal{H}^0) \) in the transformed variables, with equilibrium examples\([10]\) ranging from the KV distribution, to the waterbag equilibrium, to thermal equilibrium, to mention a few examples. Once the functional form of \( F_b^0(\mathcal{H}^0) \) is specified, and \( \psi^0 \) is calculated self-consistently from Eq. (13), periodically-focused equilibrium properties in the laboratory coordinates, such as the density profile and the transverse temperature profile, can then be determined by the pull-back transformation. For example, to the leading order, the density profile is of the form\([10]\)

\[
n_b(x, y, s) = n_b^0 \left( \frac{x}{1 - \beta_q(s)}, \frac{y}{1 + \beta_q(s)} \right),
\]

where \( n_b^0(\tilde{X}, \tilde{Y}) = \int d\tilde{X}' d\tilde{Y}' F_b^0(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}') \).

4 CONCLUSIONS

To summarize, the formalism developed here represents a powerful framework for investigating the kinetic equilibrium and stability properties of an intense nonneutral ion beam propagating through an alternating-gradient quadrupole field. First, the analysis applies to a broad class of equilibrium distributions \( F_b^0(\mathcal{H}^0) \) in the transformed variables. Second, the determination of (periodically-focused) beam properties in the laboratory frame is relatively straightforward. Third, the analysis applies to beams with arbitrary space-charge intensity, consistent only with requirement for radial confinement of the beam particles by the applied focusing field \( (\kappa_{f_q}\beta_b^2 c^2 > \gamma^2 \beta_{pb}^2 / 2\gamma_b^2) \). Finally, the formalism can be extended\([10]\) in a straightforward manner to the case of a periodic-focus solenoidal field \( B_{sol}(x) = B_z(s)e_z - (1/2)B'_z(s)(xe_x + ye_y) \), and to the case where weak nonlinear corrections to the focusing force are retained in the analysis.

5 ACKNOWLEDGEMENT

This research was supported by the Department of Energy.

6 REFERENCES