Abstract

The Beam Equilibrium Stability and Transport (BEST) code, a 3D multispecies nonlinear perturbative particle simulation code, has been developed to study collective effects in intense charged particle beams described self-consistently by the Vlasov-Maxwell equations. This code provides an effective numerical tool to investigate collective instabilities, periodically-focused beam propagation in alternating-gradient focusing fields, halo formation, and other important nonlinear process in intense beam propagation.

1 INTRODUCTION AND THEORETICAL MODEL

For accelerator applications to spallation neutron sources, tritium production, and heavy ion fusion, space-charge effects on beam equilibrium, stability, and transport properties become increasingly important. To understand these collective processes at high beam intensity, it is necessary to treat the nonlinear beam dynamics self-consistently using the nonlinear Vlasov-Maxwell equations. Recently, the δf formalism, a low-noise, nonlinear perturbative particle simulation technique, has been developed for intense beam applications, and applied to matched-beam propagation in a periodic focusing field and five-dimensional phase space.

\begin{align*}
\{ \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla - \gamma_j m_j \omega_{\beta_j} \mathbf{x}_\perp \},
\end{align*}

where \( \mathbf{x}_\perp = \mathbf{x} + y \mathbf{e}_y \) is the transverse displacement, and \( \omega_{\beta_j} \) is the effective betatron frequency for transverse oscillation. For example, in the absence of background ions (\( j = i \)), we normally assume \( V_e = \beta_i c \approx 0 \). The space-charge intensity is allowed to be arbitrarily large, subject only to transverse confinement of the beam ions by the applied focusing force, and the background electrons are confined in the transverse plane by the space-charge potential \( \phi(x, t) \) due to the excess ion charge. In the electrostatic approximation, we represent the self-electric and self-magnetic fields by \( E^e = -\nabla \phi(x, t) \) and \( B^e = \nabla \times A^e(x, t) \mathbf{e}_z \), respectively. For present purpose, assuming perturbations with long axial wavelength (\( k_x^2 r_b^2 \ll 1 \)) and neglecting the perturbed axial force on the charge components, the nonlinear Vlasov-Maxwell equation in the five-dimensional phase space \( (x, \mathbf{p}_\perp) \) can be approximated by

\begin{align*}
\left\{ \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla - \gamma_j m_j \omega_{\beta_j} \mathbf{x}_\perp \right. \\
\left. + e_j \nabla_\perp(\phi - \beta_j A_z) \right\}[\frac{\partial}{\partial \mathbf{p}_\perp}] \right\} f_j(x, \mathbf{p}_\perp, t) = 0,
\end{align*}

and

\begin{align*}
\nabla^2_\perp \phi = -4\pi \int d^2 p f_j(x, \mathbf{p}_\perp, t),
\end{align*}

\begin{align*}
\nabla^2_\perp A_z = -4\pi \int d^2 p f_j(x, \mathbf{p}_\perp, t),
\end{align*}

2 NONLINEAR δF SIMULATION METHOD AND THE BEST CODE

In the nonlinear δf formalism, we express the total distribution function as \( f_j = f_{j0} + \delta f_j \), where \( f_{j0} \) is a known solution to the nonlinear Vlasov-Maxwell equations (2) and
(3), and determine the detailed evolution of the perturbed distribution function \( \delta f_j \equiv f_j - f_{j0} \). This is accomplished by advancing the weight function defined by \( w_j \equiv \delta f_j / f_{j0} \), together with the particles’ positions and momenta. The equations of motion for the particles, obtained from the characteristics of the nonlinear Vlasov equation (2), are given by

\[
\frac{dx_{ji}}{dt} = V_j e_z + (\gamma_j m_j)^{-1} p_{ji}, \\
\frac{dp_{ji}}{dt} = -\gamma_j m_j e_β x_{ji} - e_j \nabla_⊥ (\phi - \beta_j A_z).
\]

Here the subscript “ji” labels the i-th simulation particle of the j-th species. The weight functions \( w_j \), as functions of phase space variables, are carried by the simulation particles, and the dynamical equations for \( w_j \) are easily derived from the definition of \( w_j \) and the Vlasov equation (2). Following the algebra in Refs. [4, 5, 6, 7], we obtain

\[
\frac{dw_{ji}}{dt} = -(1 - w_{ji}) \frac{\partial f_{j0}}{\partial p_{ji}} \cdot \delta \left( \frac{dp_{ji}}{dt} \right), \\
\delta \left( \frac{dp_{ji}}{dt} \right) = \frac{dp_{ji}}{dt} \bigg|_{(ϕ, A_z) → (δϕ, δA_z)},
\]

where \( δϕ = ϕ - ϕ_0 \) and \( δA_z = A_z - A_{z0} \). Here, the equilibrium solutions \( (ϕ_0, A_{z0}, f_{j0}) \) solve the steady-state \((∂/∂t = 0)\) Vlasov-Maxwell equations (2) and (3) with \( ∂/∂r = 0 \) and \( ∂/∂θ = 0 \). A wide variety of axisymmetric equilibrium solutions to Eqs. (2) and (3) have been investigated in the literature. The perturbed distribution \( \delta f_j \) is obtained through the weighted Klimontovich representation

\[
\delta f_j = \frac{N_j}{N_{sj}} \sum_{i=1}^{N_{sj}} w_{ji} \delta(x - x_{ji}) \delta(p_{⊥} - p_{⊥ji}),
\]

where \( N_j \) is the total number of actual j-th species particles, and \( N_{sj} \) is the total number of simulation particles for the j-th species. Maxwell’s equations are also expressed in terms of the perturbed fields and perturbed density according to

\[
\nabla_⊥^2 δϕ = -4π \sum_j e_j δn_j, \\
\nabla_⊥^2 δA_z = -4π \sum_j e_j β_j δn_j, \\
\delta n_j = \int d^2p δf_j(x, p_{⊥}, t) = \frac{N_j}{N_{sj}} \sum_{i=1}^{N_{sj}} w_{ji} U(x, x_{ji}).
\]

Here, \( U(x, x_{ji}) \) represents the method of distributing particles on the grids in configuration space. The nonlinear particle simulations are carried out by iteratively advancing the particle motions, including the weights they carry, according to Eqs. (4) and (5), and updating the fields by solving the perturbed Maxwell’s equations (7) with appropriate boundary conditions at the cylindrical conducting wall. Even though it is a perturbative approach, the \( \delta f \) method is fully nonlinear and simulates the original nonlinear Vlasov-Maxwell equations. Compared with conventional particle-in-cell simulations, the noise level in \( \delta f \) simulations is significantly reduced. In addition, the \( \delta f \) method can be used to study linear stability properties provided the factor \((1 - w_{ji})\) in Eq. (5) is approximated by 1, and the forcing term in Eq. (4) is replaced by the unperturbed force (i.e., advancing particles along the unperturbed orbits). Implementation of the 3D multispecies nonlinear \( \delta f \) simulation method described above is embodied in the BEST code at the Princeton Plasma Physics Laboratory. The code advances the particle motions using a 4th-order Runge-Kutta method, and solves Maxwell’s equations by a fast Fourier transform and finite-difference method in cylindrical geometry. Written in Fortran 90/95, the code utilizes extensively the object-oriented features provided by the computer language. The NetCDF scientific data format is implemented for large-scale diagnostics and visualization. The code has achieved an average speed of 40µs/(particle × step) on a DEC alpha personal workstation 500au computer.

3 SIMULATION RESULTS

For brevity, we present here illustrative simulation results for a single-species thermal equilibrium ion beam in a constant focusing field. In this case, equilibrium properties depend on the radial coordinate \( r = (x^2 + y^2)^{1/2} \). The thermal equilibrium distribution function is given by

\[
f_{j0}(r, p_{⊥}) = \frac{\hat{n}_b}{2\pi γ_b m_b T_b} \times \exp \left\{ -\frac{p_{⊥}^2}{2γ_b m_b} - \gamma_b m_w ω_b^2 r^2 / \left(2 + e_b(ϕ_0 - β_b A_{z0})\right) \right\}
\]

where \( \hat{n}_b \) is the density of beam particles at \( r = 0 \), and \( T_b \) is the transverse temperature of the beam ions in energy units. It is also assumed that the beam is centered inside a cylindrical chamber with perfectly conducting wall located at \( r = r_w \). The equilibrium self-field potentials \( ϕ_0 \) and \( A_{z0} \) can be determined numerically from Maxwell’s equations (3). First, we examine the nonlinear propagation properties of the beam. A random initial perturbation is introduced into the system, and the beam is propagated from \( t = 0 \) to \( t = 500τ_β \), where \( τ_β = ω_β^{-1} \). The simulation results show that the perturbations do not grow and the beam propagates quiescently, which agrees with the nonlinear stability theorem[14, 15] for the choice of equilibrium distribution function in Eq. (8). Shown in Fig. 1 is a plot of the change in transverse emittance-squared (normalized by \( ω^2_b/ω^2_b \)), \( δε^2 = ε^2(t) - ε^2_0 \) versus normalized time \( t/τ_β \), for perturbations about the thermal equilibrium distribution in Eq. (8). The system parameters in Fig. 1 correspond to protons with \( γ_b = 1.85 \), and normalized beam intensity \( Kβγeτβ/ε_0 = 0.025 \), where \( K = 2N_b e^2/γ_b^2 m_b ω_b^2 e^2 \) is the self-field perveance, and \( N_b \) is the number of beam ions per
unit axial length. The amplitudes of the initial random perturbation in weights in Fig. 1 is $10^{-4}$, which leads to the very small offset in $\delta e^2$. It is evident from Fig. 1 that the variations in beam emittance, $\delta e^2$, remain extremely small for perturbations about a thermal equilibrium beam. As a

second example, we study the linear surface mode for perturbations about a thermal equilibrium beam in the space-charge-dominated regime, with flat-top density profile and $K_s b c r_\beta / e_0 \gg 1$. These modes are of practical interest because they can be destabilized by a two-stream electron-ion interaction when background electrons are present [9, 10]. The BEST code, operating in its linear stability mode, has recovered very well-defined eigenmodes with mode structures and eigenfrequencies which agree well with theoretical predications. For $K_s b c r_\beta / e_0 \gg 1$, and azimuthal mode number $l = 1$, the dispersion relation for these modes is given by [1, 9, 10]

$$\omega = k_z V_b \pm \frac{\tilde{\omega}_{pb}}{\sqrt{2} \gamma_b} \sqrt{1 - \frac{r_b^2}{r_w^2}},$$

(9)

where $r_b$ is the radius of the beam edge, and $r_w$ is location of the conducting wall. In Eq. (9), $\tilde{\omega}_{pb}^2 = 4\pi n_b e^2 / \gamma_b m_b$ is the ion plasma frequency-squared, and $\tilde{\omega}_{pb}/\sqrt{2} \gamma_b \approx \omega_{pb}$ in the space-charge-dominated limit. Shown in Fig. 2 is the comparison between plots of the eigenfrequency versus $r_w/r_b$ obtained from the simulations (diamonds and triangles) and that predicted by Eq. (9) (solid curves). The parameters for this case are chosen close to the space-charge limit with $K_s b c r_\beta / e_0 = 6.59$, and the perturbation has normalized axial wavenumber $k_z V_b / \omega_{pb} = 2\pi$. It is clear from Fig. 2 that the simulation results agree well with theory.

Figure 2: Eigenfrequency versus $r_w/r_b$.

4 CONCLUSION AND FUTURE WORK

The BEST code has been tested and applied in different scenarios. As a 3D multispecies perturbative particle simulation code, it provides several unique capabilities. Since the simulation particles are used to simulate only the perturbed distribution function and self-fields, the simulation noise is reduced significantly. The perturbative approach also enables the code to investigate different physics effects separately, as well as simultaneously. The code can be easily switched between linear and nonlinear operation, and used to study both linear stability properties and nonlinear beam dynamics. These features, combined with 3D and multispecies capabilities, provide us with an effective tool to investigate the electron-ion two-stream instability, periodically focused solutions in alternating focusing fields, halo formation, and many other important problems in nonlinear beam dynamics and accelerator physics. Finally, the BEST code is readily adapted to the case where the applied focusing force, $F_j^{\text{loc}}$, corresponds to a periodic focusing quadrapole field or solenoidal field, and the effects of the axial self-field field $F_j^{\text{ax}} = -e_j c_j \partial \phi(x,t)/\partial z$ on the particle dynamics are retained self-consistently. Results of these studies will be reported in future publications.

5 ACKNOWLEDGMENT

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6 REFERENCES


[8] Alex Friedman, private communication.


