NONLINEAR ACCELERATOR PROBLEMS VIA WAVELETS: 7. INVARIANT CALCULATIONS IN HAMILTON PROBLEMS

A. Fedorova, M. Zeitlin, IPME, RAS, St. Petersburg, Russia *

Abstract

In this series of eight papers we present the applications of methods from wavelet analysis to polynomial approximations for a number of accelerator physics problems. In this paper we consider invariant formulation of nonlinear (Lagrangian or Hamiltonian) dynamics on semidirect structure (relativity or dynamical groups) and corresponding invariant calculations via CWT.

1 INTRODUCTION

This is the seventh part of our eight presentations in which we consider applications of methods from wavelet analysis to nonlinear accelerator physics problems. This is a continuation of our results from [1]-[8], in which we considered the applications of a number of analytical methods from nonlinear (local) Fourier analysis, or wavelet analysis, to nonlinear accelerator physics problems both general and with additional structures (Hamiltonian, symplectic or quasicomplex), chaotic, quasiclassical, quantum. Wavelet analysis is a relatively new set of mathematical methods, which gives us a possibility to work with well-localized bases in functional spaces and with the general type of operators (differential, integral, pseudodifferential) in such bases. In contrast with parts 1–4 in parts 5–8 we try to take into account before using power analytical approaches underlying geometrical, topological structures related to kinematical, dynamical and hidden symmetry of physical problems. We described a number of concrete problems in parts 1–4. The most interesting case is the dynamics of spin-orbital motion (part 4). In section 2 we consider dynamical consequences of covariance properties regarding to relativity (kinematical) groups and continuous wavelet transform (CWT) (in section 3) as a method for the solution of dynamical problems. We introduce the semidirect product structure, which allows us to consider from general point of view all relativity groups such as Euclidean, Galilei, Poincare. Then we consider applications of methods from wavelet analysis to polynomial approximations for a number of accelerator physics problems. In this series of eight papers we present the applications of methods from wavelet analysis to polynomial approximations for a number of accelerator physics problems. In this paper we consider invariant formulation of nonlinear (Lagrangian or Hamiltonian) dynamics on semidirect structure (relativity or dynamical groups) and corresponding invariant calculations via CWT.

2 DYNAMICS ON SEMIDIRECT PRODUCTS

Relativity groups such as Euclidean, Galilei or Poincare groups are the particular cases of semidirect product construction, which is very useful and simple general construction in the group theory [9]. We may consider as a basic example the Euclidean group $SE(3) = SO(3) \rtimes \mathbb{R}^3$, the semidirect product of rotations and translations. In general case we have $S = G \rtimes V$, where group $G$ (Lie group or automorphisms group) acts on a vector space $V$ and on its dual $V^*$. Let $V$ be a vector space and $G$ is the Lie group, which acts on the left by linear maps on $V$ (G also acts on the left on its dual space $V^*$). The Lie algebra of $S$ is the semidirect product Lie algebra, $s = G \rtimes V$ with brackets $[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], [\xi_1, v_2] - \xi_2 v_1)$, where the induced action of $G$ by concatenation is denoted as $\xi_1 v_2$. Let $(g, v) \in S = G \times V$, $(\xi, u) \in s = G \times V$, $(\mu, a) \in s^* = G^* \times V^*$, $g\xi = Ad_g \xi, g\mu = Ad_g \mu, ga$ denote the induced left action of $g$ on $a$ (the left action of $G$ on $V$ induces a left action on $V^*$ — the inverse of the transpose of the action on V), $\rho_v : G \to V$ is a linear map given by $\rho_v(\xi) = \xi v$, $\rho_{v^*} : V^* \to G^*$ is its dual. Then adjoint and coadjoint actions are given by simple concatenation: $(g, v)(\xi, u) = (g\xi, g\mu - (g\xi) v), (g, v)(\mu, a) = (g\mu + \rho_v^*(ga), ga)$. Also, let be $\rho_{v^*} a = v \circ a \in G^*$ for $a \in V^*$, which is a bilinear operation in $v$ and $a$. So, we have the coadjoint action: $(g, v)(\mu, a) = (g\mu + v \circ (ga), ga)$. Using concatenation notation for Lie algebra actions we have alternative definition of $v \circ a \in G^*$. For all $v \in V$, $a \in V^*$, $\eta \in G$ we have $\langle \eta a, v \rangle = - \langle v \circ a, \eta \rangle$.

Now we consider the manifestation of semidirect structure of symmetry group on dynamical level. Let $\mathcal{F}, G \mathcal{B}$ be real valued functions on the dual space $G^*$, $\mu \in G^*$. Functional derivative of $F$ at $\mu$ is the unique ele-
We consider the case of the left representation and the left corresponding Legendre transformation is a diffeomorphism. The Lie-Poisson equations, determined by \( \mathcal{F} = \{ F, H \} \) or intrinsically \( \dot{\mu} = \mp ad^*_{\delta H/\delta \mu} \), for the left representation of G on V is Lie-Poisson brackets on V with initial conditions on the level of Euler-Poincare equations.

Then we have four equivalent descriptions of the corresponding dynamics: 1. If \( a_0 \) is fixed then Hamilton’s variational principle \( \delta \int_0^t \mathcal{L}_{a_0}(g(t), \dot{g}(t)) dt = 0 \) holds for variations \( \delta g(t) \) of \( g(t) \) vanishing at the endpoints. 2. \( g(t) \) satisfies the Euler-Lagrange equations for \( L_{a_0} \) on G. 3. The constrained variational principle \( \delta \int_0^t \mathcal{L}^t(\xi(t), a(t)) dt = 0 \) holds on \( G \times V^* \), using variations of \( \xi \) and \( a \)

So, we may apply our wavelet methods either on the level of variational formulation or on the level of Euler-Poincare equations.

### 3 CONTINUOUS WAVELET TRANSFORM

Now we need take into account the Hamiltonian or Lagrangian structures related with systems (2) or (3). Therefore, we need to consider generalized wavelets, which allow us to consider the corresponding structures instead of compactly supported wavelet representation from parts 1-4. In wavelet analysis the following three concepts are used now: 1) a square integrable representation \( U \) of a group \( G \), 2) coherent states (CS) over \( G \), 3) the wavelet transform associated to U. We consider now their unification [10]-[12]. Let \( G \) be a locally compact group and \( U_a \) strongly continuous, irreducible, unitary representation of \( G \) on Hilbert space \( \mathcal{H} \). Let \( H \) be a closed subgroup of \( G \), \( X = G/H \) with (quasi) invariant measure \( \nu \) and \( \sigma : X = G/H \to G \) is a Borel section in a principal bundle \( G \to G/H \). Then we say that \( U \) is square integrable mod\( (H, \sigma) \) if there exists a non-zero vector \( \eta \in \mathcal{H} \) such that \( 0 < \int_X | <U(\sigma(x)) \eta| \Phi >|^2 \nu(x) = <\Phi | A_\eta \Phi > < \infty, \forall \Phi \in \mathcal{H} \). Given such a vector \( \eta \in \mathcal{H} \) called admissible for \((U, \sigma)\) we define the family of (covariant) coherent states or wavelets, indexed by points \( x \in X \), as the orbit of \( \eta \) under \( G \), though the representation \( U \) and the section \( \sigma \)

### 3.1 OVERCOMPLETENESS

The set \( \mathcal{G} \) is total in \( \mathcal{H} : (\mathcal{G})^\perp = 0 \). Solution property: the square integrability condition may be represented as a resolution relation: \( \int_X | <\eta(\sigma(x)) | \Phi >|^2 \nu(x) = <\Phi | A_\eta \Phi > < \infty, \forall \Phi \in \mathcal{H} \). Given such a vector \( \eta \in \mathcal{H} \) called admissible for \((U, \sigma)\) we define the family of (covariant) coherent states or wavelets, indexed by points \( x \in X \), as the orbit of \( \eta \) under \( G \), though the representation \( U \) and the section \( \sigma \)

Thus, we have four equivalent descriptions of the corresponding dynamics: 1. If \( a_0 \) is fixed then Hamilton’s variational principle \( \delta \int_0^t \mathcal{L}_{a_0}(g(t), \dot{g}(t)) dt = 0 \) holds for variations \( \delta g(t) \) of \( g(t) \) vanishing at the endpoints. 2. \( g(t) \) satisfies the Euler-Lagrange equations for \( L_{a_0} \) on G. 3. The constrained variational principle \( \delta \int_0^t \mathcal{L}^t(\xi(t), a(t)) dt = 0 \) holds on \( G \times V^* \), using variations of \( \xi \) and \( a \) of the form \( \delta \xi = \dot{\xi} + [\xi, \eta] \), \( \delta a = -\eta \), where \( \eta \in \mathcal{G} \) vanishes at the endpoints. 4. The Euler-Poincare equations hold on \( G \times V^* \).

\[
\frac{d}{dt} \xi = \mp \delta \mathcal{L} / \delta \dot{\xi} = a \delta \mathcal{L} / \delta \dot{\xi} + \dot{\xi} \delta \mathcal{L} / \delta \xi \quad (3)
\]

So, we may apply our wavelet methods either on the level of variational formulation or on the level of Euler-Poincare equations.
\[ \Phi(x) = \int_X K_\eta(x,y) \Phi(y) \, d\nu(y), \quad \forall \Phi \in \mathcal{H}_\eta. \] The kernel is given explicitly by \( K_\eta(x,y) = \eta_{\eta(x)} A_{\eta(x)}^{-1} \eta(y), \) if \( \eta(y) \in D(A_{-1}^{-1}), \forall y \in X. \) So, the function \( \Phi \in L^2(X, d\nu) \) is a wavelet transform (WT) if it satisfies this reproducing relation. 4. Reconstruction formula. The WT \( W_\eta \) may be inverted on its range by the adjoint operator, \( W_\eta^{-1} = W_{\eta^*}^* \) on \( \mathcal{H}_\eta \) to obtain for \( \eta_{\eta(x)} \in D(A_{-1}^{-1}), \forall x \in X \) \( W_\eta^{-1} \Phi = \int_X \Phi(x) A_{\eta(x)}^{-1} \eta_{\eta(x)}(x) \, d\nu(x), \quad \Phi \in \mathcal{H}_\eta. \) This is inverse WT. If \( A_{-1}^{-1} \) is bounded then \( S_\eta \) is called a frame, if \( \Lambda_\eta = \lambda \) then \( S_\eta \) is called a tight frame. This two cases are generalization of a simple case, when \( S_\eta \) is an (ortho)basis.

The most simple cases of this construction are:

1. \( H = \{ e \} \). This is the standard construction of WT over a locally compact group. It should be noted that the square integrability of \( U \) is equivalent to \( U \) belonging to the discrete series. The most simple example is related to the affine \((ax+b)\) group and yields the usual one-dimensional wavelet analysis \( \left[ \tau(b,a)f(x) \right](x) = \frac{1}{\sqrt{|a|}} f \left( \frac{x-b}{a} \right). \) For \( G = SIM(2) = \mathbb{R}^2 \cong (\mathbb{R}^2 \times SO(2)), \) the similitude group of the plane, we have the corresponding two-dimensional wavelets.
2. \( H = H_\eta \), the isotropy (up to a phase) subgroup of \( \eta \): this is the case of the Gilmore-Peregomov CS. Some cases of group \( G \) are:
   a. Semisimple groups, such as \( SU(N), SU(N)M, SU(p,q), Sp(N,R). \)
   b. the Weyl-Heisenberg group \( G_{WH} \) which leads to the Gabor functions, i.e. canonical (oscillator)coherent states associated with windowed Fourier transform or Gabor transform (see also part 6): \( \left[ \tau(q,p,\varphi)f(x) \right](x) = \exp(iq(\varphi-p(x-q))) \, f(x-q). \) In this case \( \mathcal{H} \) is the center of \( G_{WH} \). In both cases time-frequency plane corresponds to the phase space of group representation.
   c. The similitude group \( SIM(n) \) of \( \mathbb{R}^n(n \geq 3): \) for \( H = SO(n-1) \) we have the axissymmetric n-dimensional wavelets.
   d. We also have the case of bigger group, containing both affine and Weyl-Heisenberg group, which interpolate between affine wavelet analysis and windowed Fourier analysis: affine Weyl-Heisenberg group [12].
   e. Relativity groups. In a nonrelativistic set-up, the natural kinematical group is the (extended) Galilei group. Also we may add independent space and time dilations and obtain affine Galilei group. If we restrict the dilations by the relation \( a_0 = a^2, \) where \( a_0, a \) are the time and space dilation we obtain the Galilei-Schrödinger group, invariance group of both Schrödinger and heat equations. We consider these examples in the next section. In the same way we may consider as kinematical group the Poincare group. When \( a_0 = a \) we have affine Poincare or Weyl-Poincare group. Some useful generalization of that affinization construction we consider for the case of hidden symplectic structure in part 6. But the usual representation is not square–integrable and must be modified: restriction of the representation to a suitable quotient space of the group (the associated phase space in our case) restores square – integrability: \( G \rightarrow \) homogeneous space. Our goal is applications of these results to problems of Hamiltonian dynamics and as consequence we need to take into account symplectic nature of our dynamical problem. Also, the symplectic and wavelet structures must be consistent (this must be resemble the symplectic or Lie-Poisson integrator theory). We use the point of view of geometric quantization theory (orbit method) instead of harmonic analysis. Because of this we can consider (a) – (e) analogously. In next part we consider construction of invariant bases.

We are very grateful to M. Cornacchia (SLAC), W. Herrmannsfeldt (SLAC) Mrs. J. Kono (LBL) and M. Laraneta (UCLA) for their permanent encouragement.