Muon Colliders - Ionization Cooling and Solenoids*

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Abstract

For a muon collider, to obtain the needed luminosity, the phase space volume must be greatly reduced within the muon life time. The ionization cooling is the preferred method used to compress the phase space and reduce the emittance to obtain high luminosity muon beams. Alternating solenoid lattices has been proposed for muon colliders, where the emittances are large. We present an overview, discuss formalism, transfer maps for solenoid magnets and beam dynamics.

1 INTRODUCTION

Alternating solenoid lattices has been proposed as desirable for use in the earlier cooling stages of Muon Colliders, where the emittances are large. Since the minimum $\beta_z$'s must decrease in order to obtain smaller transverse emittances as the muon beam travels down the cooling channel. This can be done by increasing the focusing fields and/or decreasing the muon momenta, where the current carrying lithium lenses may be used (to get a stronger radial focusing and to minimize the final emittance) for the last few cooling stages. The use of 'bent solenoids' may provide the required dispersion for the momentum measurement. Where the off-momentum muons are displaced vertically by an amount:

$$\Delta y \approx \frac{P}{eB_s} \Delta P \theta_{\text{bend}},$$

where $B_s$ is the field of the bent solenoid and $\theta_{\text{bend}}$ is the bend angle. In Fig.1, the bending of the solenoid produce the dispersion required for the longitudinal to transverse emittance exchange. Where after one bend and one set of wedges the beam cross-section is asymmetric then the symmetry is restored by going through the second bend and wedge system (which is rotated by 90 degrees w.r.t. the first) [1].

2 FORMULATION AND MAPS FOR SOLENOIDS

The canonical equations in 2n-Dimensional phase space (e.g. 6 Dim., in our calculation) can be expressed as $\frac{d\psi_i}{dt} = [\psi_i, H], i = 1, 2, \ldots 2n$, and in terms of the Lie transformations as

$$\frac{d\psi_i}{dt} = - : H : \psi_i, \quad i = 1, 2, \ldots 2n \quad (2)$$

Where the Lie operator (: $H :$) is generated by the Hamiltonian, $(H)$, and Lie transformation, $M = e^{-t:H}$, could generate the solution to Eq. (2) as $\psi_i = M\psi_i(0)$, where $\psi_i$ is the value of $\psi_i(t)$ at $t > 0$ and $\psi_i(0)$ is the initial trajectory. The interest is to find solutions to equations of motion which differ slightly from the reference orbit. Thus, one can choose the canonical variables, from the values for the reference trajectory (for small deviations) and Taylor expand the Hamiltonian $(H)$ about the design trajectory $H = H_2 + H_3 + \ldots$. Where $H_n$ is a homogeneous polynomial of degree n in the canonical variables. After transformations to the normalized dimensionless variables, one can obtain the effective Hamiltonian $H^{\text{New}}$, expressed as

$$H^{\text{New}} = F_2 + F_3 + F_4 + \ldots \quad (3)$$

Thus the particle trajectory $\psi^* = (X, P_X, Y, P_Y, \tau, P_\tau)$ through a beamline element of length $L$ can be described by $\psi^*_i = - : H^{\text{New}} : \psi_i, i = 1, 2, \ldots 2n$. The exact symplectic map that generates the particle trajectory through that element is $M = e^{-L:H^{\text{New}}}$, where $M$ describes the particle behavior through the element of length $L$. Using the factorization and expanding $H^{\text{New}}$ as in Eq. (3), results in

$$M = e^{-L:H^{\text{New}}} = e^{-f_2}; e^{-f_3}; e^{-f_4}; \ldots\quad (4)$$

(e.g., for a map through 3rd order we need to include terms of $f_2, f_3$, and $f_4$).

To illustrate the above formalism, consider the evolution of the motion of particles in an external electromagnetic field described by the Hamiltonian $H = \sqrt{m^2c^4 + e^2[(p_x - qA_x)^2 + (p_y - qA_y)^2 + (p_z - qA_z)^2]} + e\phi(x, y, z; t)$, where $m$ and $q$ are the rest mass and charge of the particle, $A$ and $\phi$ are the vector and scalar potentials such that $\vec{B} = \nabla \times \vec{A}, \vec{E} = -\nabla \phi - \nabla \vec{A}/\Omega t$.

Making a canonical transformation from $H$ to $H_1$ and changing the independent variable from time $t$ to $z$ (for convenience) for a particle in magnetic field (e.g. of solenoid) results in $p_z = [(p_x - qA_x)^2 + (p_y - qA_y)^2 + p_z^2/e^2 - m^2c^2]^{1/2}$. Where $H = -p_z$, $H_1 = -p_z$ and $t = (z/v_0)$ the time as a function of $z$. We now make a
canonical transformation from \( H_1 \) to \( H^{\text{New}} \), with a dimensionless deviation variables (for convenience), \( X = x/l, \ Y = y/l, \ \tau = c/(l-t/v_{\text{eq}}) \), \( P_x = p_x/p_0, \ P_y = p_y/p_0, \ P_z = (p_z - \rho_0\tau)/\rho_0c \), where \( l \) is a length scale (taken as 1 m in our analysis), with \( \mathbf{P} = \tilde{P}_x + \tilde{P}_y \) and \( \mathbf{Q} = \tilde{X} + \tilde{Y} \) defined as two dimensional vectors \([5], p_0 \) and \( \rho_0 \) are momentum and energy scales. Where \( p_0 \) is the design momentum, \( \rho_0 \) is the velocity on the design orbit and \( \rho_0^2 \) is a value of \( p_t \) on the design orbit \((p_0 = \sqrt{m^2c^4 + p_t^2c^2})\) (reminding that design orbit for the solenoid is along the \( z \)-axis). Thus, expanding the new Hamiltonian Eq. (3) leads to:

\[
F_2 = \frac{P_z^2}{(2\beta^2\gamma^2)} - \frac{1}{2}B_0(\tilde{Q} \times \tilde{P}) \cdot \frac{1}{8}B_0^2Q^2 + \frac{P_e^2}{2} \tag{5}
\]

\[
F_3 = \frac{P_z^3}{(2\beta^2\gamma^2)} - \frac{P_z}{2\beta}B_0(\tilde{Q} \times \tilde{P}) \cdot \tilde{z} + \frac{P_z^2}{8\beta^3}B_0^2Q^2 + 4P^2 \tag{6}
\]

\[
F_4 = \frac{P_z^4(5 - \beta^2)}{8\beta^4\gamma^2} + \frac{P_z^2Q^2B_0^2(3 - \beta^2)}{16\beta^2} + \frac{P_z^2}{2}\tilde{Q} \times \tilde{P} \cdot \tilde{z}B_0(3 - \beta^2) \frac{2}{\beta^2} + \frac{P_z^2}{2}P_0^2(3 - \beta^2) + \frac{Q^4}{16}(B_0^4 - 4B_0B_2)/8 + \frac{Q^2}{4}P_0^23B_0^2/4 + \frac{Q^2}{4}(\tilde{Q} \times \tilde{P}) \cdot \tilde{z}(B_0 - B_0^4)/4 - \frac{1}{8}(\tilde{P} \cdot \tilde{Q})^2B_0 - \frac{P_e^2}{8\beta^3}B_0^2 + \frac{P_e^4}{8} \tag{7}
\]

Following the Hamiltonian flow generated by:

\[ H^{\text{New}} = F_2 + F_3 + \ldots \]

from some initial \( \psi_0 \) to a final \( \psi_f \) coordinates we can calculate the transfer map \( M \) (Eq. (4)) for the solenoid. Where \( F_2, F_3, \) and \( F_4 \) would lead to the 1st, 2nd, and 3rd order maps. The effects of which can be seen from Eqs. (5–7). For example, the 2nd order effects due to solenoid transfer maps are purely chromatic aberrations Eq. (6). In addition, we note the third order geometric aberrations Eq. (7). As shown by Eqs. (5–7), the coupling between \( X, Y \) planes produced by a solenoid is rotation about the \( z \)-axis which is a consequence of rotational invariance of the Hamiltonian \( H^{\text{New}} \), due to axial symmetry of the solenoid field. For beam simulations, \( M \) can be calculated to any order using numerical integration techniques such as Runge-Kutta method depending on the computer memory and space available \([5]\).

3. HIGHER ORDER KINEMATIC INVARIANTS AND CORRELATIONS:

Let \( \rho(\psi) \) be the distribution of particles in phase space at any instant e.g. \( d^3N = \rho(\psi)d^3\psi \), where \( d^3N \) and \( d^3\psi \) are the number of particles, and small volume in the 6-dimensional phase space \((\psi = [\hat{q}, \hat{p}, q_i, p_i = 1, 2, 3], \) respectively. Let \( \rho(\mathcal{M}^{-1}) \) be the final distribution at the end of the system such that a set of initial moments are \((j = \text{index})\), defined as \( k_j' = \int \rho(\psi)F_j(\psi)d\psi \). Where the final moments become \( k_j' = \int \rho(\psi)f_j(\mathcal{M}\psi')d\psi' \) with \( F_j(\mathcal{M}) = \sum D_{jk}(\mathcal{M})F_j(\psi) \), \( (D_{jk} \) is a matrix and \( F_j(\psi) \) are a complete set of homogeneous polynomials.) Thus, the moment transport can be expressed in a simple form as \( k_j' = \sum F_j(\mathcal{M})k_j0 \). \( D_{jk}(\mathcal{M}) \) are quadratic functions of matrix elements \( M_{ij} \).

In 6-Dim. phase space, there are 3 functionally independent kinematic invariants made up of quadrature moments, e.g. \( e_x, e_y, e_z \), such that \( I_2(k) = e_x^2 + e_y^2 + e_z^2 \), \( I_4(k) = e_x^4 + e_y^4 + e_z^4 \), and \( I_6(k) = e_x^6 + e_y^6 + e_z^6 \), or in general \( I_k(k) = \frac{1}{2}(-1)^{k/2}t_0(\psi)^2 \), where \( \psi = 6 \times 6 \) matrix, whose entries are moments, \( \psi_{jk} = \langle \psi_j\psi_k \rangle \), with \( I = 3 \times 3 \) identity matrix and \( J = - \left[ \begin{array}{c c c} 0 & I \end{array} \right) \).

E.G., \( I_2(k) = \langle x^2 \rangle \langle p_x^2 \rangle - \langle x^2 \rangle \langle p_x \rangle^2 - \langle p_x \rangle^2 \langle x \rangle^2 - 2\langle xp_x \rangle^2 \). This is a generalization of 2-Dim. mean square emittance (e.g. see Refs. \([2, 5]\)). Thus, higher order kinematic invariants (e.g. cubic and quartic moments); and correlations between various degrees of freedom may be constructed, and used as a tool, in nonlinear dynamic studies. E.G., for a beam transport system one may use an invariant: \( I \equiv \langle x^2 \rangle \langle p_x^2 \rangle + \langle p_x^2 \rangle \langle x \rangle^2 - 2\langle xp_x \rangle^2 \) constructed from a linear and quadratic moments. Noting that, the inclusion of correlations between the variables may be detrimental (in the accuracy of) beam dynamic studies.

4. MUON COOLING

Muon colliders have the potential, to provide a probe for fundamental particle physics. To obtain the needed collider luminosity, the phase-space volume must be greatly reduced within the muon life time. The Ionization cooling is the preferred method used to compress the phase space and reduce the emittance to obtain high luminosity muon beams. We note that, the ionization losses results not only in damping, but also heating: transverse heating appears due to multiple Coulomb scattering and longitudinal one is due to so named “straggling” of the ionization losses (we note that, this straggling is produced by fast “knock-on” ionization electrons), e.g. see \([4]\). The longitudinal muon momentum is then restored by coherent re-acceleration, leaving a net loss of transverse momentum (transverse cooling). To achieve a large cooling factor the process is repeated many times. The transverse cooling can be expressed (neglecting correlations) as

\[
\frac{dn}{ds} = \frac{1}{\beta^2} \frac{dE_{\mu}}{ds} \frac{\epsilon_n}{E_{\mu}} + \frac{1}{\beta^2} \frac{\epsilon_n}{E_{\mu}} \frac{1}{2\nu} \frac{m_{\mu}}{L_{R}} + \ldots \tag{8}
\]

where \( \beta = v/c, \ \epsilon_n \) is the normalized emittance, \( \beta_{\perp} \) is the betatron function at the absorber, \( dE_{\mu}/ds \) is the energy loss, and \( L_{R} \) is the radiation length of the material. The first term in this equation is the cooling term, and the second is the heating term due to multiple scattering. To minimize the heating term, a strong-focusing (small \( \beta_{\perp} \)) and a low-Z absorber (large \( L_{R} \)) is needed.

In obtaining Eq. 8, the correlations were neglected (as e.g. in the Status Report see \([1]\)), e.g. \( \langle xP_x \rangle = 0 \), and the relation \( \langle x^2 \rangle = \epsilon_{\beta_{\perp}} = \frac{\epsilon_{\beta_{\perp}}}{\beta_{\perp}} \) was used, which can not be
(e.g. Kinetic energy 10 to 150 KeV), and the damping rate is difficult to use for high energy muon source, (in addition to big noises due to coulomb scattering etc.). Classical Ionization Cooling is useable for kinetic energy range of 30 to 100 MeV. Which due to absence of “natural” longitudinal cooling it is necessary to use “wedges” for which R & D is needed. A proposal for such studies is being considered [1].

5 MUON COOLING “MERIT FACTOR”

Luminosity of collider $L$ is defined by the following expression:

$$L \sim \frac{N^2 f^{g_2 y}}{\epsilon_f} = \frac{N^2 f^{g_2 y}}{\epsilon_f}$$

(11)

Where $N$ is a number of muons per bunch, $f$ is mean repetition frequency of collisions, $\epsilon_f$ is emittance at collision point and $\beta_1 f^{g_2 y}$ is limited by condition: $\beta_1 f^{g_2 y} \geq \sigma_1 f^{g_2 y}$ where $\sigma_1$ is a longitudinal bunch size. Let us assume, that: 1) $\Delta p / N$ is known (monochromatic experiments); 2) we can redistribute emittances inside a given six-dimensional phase volume. Then, taking into account losses in the cooling system, we can rewrite Eq. (11) in the following form:

$$L \sim \frac{N_0^* \exp \left(- \frac{2}{e \varepsilon_f} \frac{\sigma_1}{\varepsilon_f} \frac{d^2}{\varepsilon_f} \right)}{\sqrt{\varepsilon_f \beta_1 f^{g_2 y}}}$$

(12)

Here “$N_0^*$” is a number of particles at an entrance of the cooling system, “$\exp$” describes muon decay, “$D^2$” describes muon losses in cooling section, and “$\varepsilon_f \beta_1 f^{g_2 y}$” is an invariant six-dimensional phase volume of muon beam.

Thus we can introduce “merit factor” which describes a quality of muon cooling system. We obtain

$$R = \frac{D^2 \exp \left[- \frac{2}{e \varepsilon_f} \frac{\sigma_1}{\varepsilon_f} \frac{d^2}{\varepsilon_f} \right]}{\sqrt{\varepsilon_f \beta_1 f^{g_2 y}}}$$

(13)

Note that, the dependence on $\varepsilon_f \beta_1 f^{g_2 y}$ may be stronger. With account of all the circumstances, we can write

$$R \sim \left(\varepsilon_f \beta_1 f^{g_2 y}\right)^\alpha$$

(14)

with $\alpha$ in interval (0.5; 2/3). For more info. see Refs. [1-5].

6 REFERENCES


