HYBRID COMPUTATION OF NORMAL MODE TUNE SHIFTS IN ROUNDED-RECTANGULAR PIPES

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Abstract
A fast and accurate hybrid (analytical-numerical) technique for computing the normal mode tune-shifts in rounded-rectangular (stadium) pipes is introduced based on Galerkin method together with a smart representation of Poisson’s equation Green’s function in a rectangular domain. Comparison with standard finite-elements and finite difference methods shows that our method is faster and more accurate, requiring no numerical differentiation.

1 THE PROBLEM
Many beam-pipe cross-section geometries of potential interest for accelerators, including the stadium-shaped one recently proposed for LHC [3], differ from the rectangle only by the rounding of corners, or the substitution of straight sides with circular arcs. Computing the related betatron tune-shifts, resulting from collective (space-charge and image) effects is a key problem to prevent resonant betatron excitations leading to potentially harmful beam instabilities. The normal mode coherent and incoherent tune-shifts can be written in terms of the normal mode Laslett coefficients \( \epsilon \) as follows [2]:

\[
\Delta \nu = -\frac{N R r_0}{\pi \nu / a_0} L^2 \epsilon, 
\]

where:

\[
\epsilon_{1,2} = \frac{L^2}{16 \Lambda} \left\{ \frac{\delta_y \partial_x \Phi^{(im)} + \delta_x \partial_y \Phi^{(im)}}{2} \right. \\
\left. + \left[ \left( \delta_y \partial_y \Phi^{(im)} - \delta_x \partial_x \Phi^{(im)} \right) \right] ^2 \right. \\
\left. + \delta_x \partial_x \Phi^{(im)} \delta_y \partial_y \Phi^{(im)} \right\}^{1/2}, 
\]

\( \Phi^{(im)} \) is the image-potential produced in the beam pipe by a linear charge density \( \Lambda \) going through the beam center of charge \( \mathbf{r}_a \), \( N \) is the number of particles in the beam, \( R \) is the machine radius, \( r_0 \) is the classical particle radius, \( L \) is a scaling length (usually, the maximum pipe diameter), \( \nu \) is the nominal tune, and

\[
\begin{cases}
\delta_{x,y} = \partial_{x,y}|_{\mathbf{r}=\mathbf{r}_a}, \text{ incoherent regime,} \\
\delta_{x,y} = (\partial_{x,y} + \partial_{x,y}|_{\mathbf{r}=\mathbf{r}_a}), \text{ coherent regime.}
\end{cases}
\]

2 THE METHOD
For computing the image potential \( \Phi^{(im)} \) in rounded rectangular geometries, it is convenient to use the rectangular-domain Green’s function \( g_R \) (henceforth RDGF), viz.:

\[
\Phi^{(im)}(\mathbf{r}, \mathbf{r}_a) = \Phi(\mathbf{r}, \mathbf{r}_a) - \Lambda g_0(\mathbf{r}, \mathbf{r}_a),
\]

\[
\Phi(\mathbf{r}, \mathbf{r}_a) = \Lambda \left[ \sum_{k} g_R(\mathbf{r}, \mathbf{r}_k) \rho_{\sigma_k}(l_k) dl_k + g_R(\mathbf{r}, \mathbf{r}_a) \right],
\]

where \( g_0 \) is the free-space Green’s function, the unknown \( \rho_{\sigma_k} \) are obviously nonzero only on the rounded portion of \( \partial S_0 \), i.e., the arcs \( \sigma_k \) and \( l_k \) is the arc-length on \( \sigma_k \).

We seek a hybrid (analytical-numerical) solution of eq. (4) by using Galerkin (moments) method [6], whereby we first expand the unknown \( \rho_{\sigma_k} \):

\[
\rho_{\sigma_k}(l_k) = \sum_{n=1}^{N} b_n^{(k)} w_n^{(k)}(l_k),
\]

into a suitable (finite) set of basis functions \( \{w_1^{(k)}(l_k), \ldots, w_N^{(k)}(l_k)\} \), defined on \( \sigma_k \), where \( \{b_1^{(k)}, \ldots, b_N^{(k)}\} \), are \( N \)-dimensional vectors of unknown coefficients, and then enforce the (Dirichlet) boundary conditions on the arcs \( \sigma_k \), whence:

\[
\int_{\sigma_k} \Phi(\mathbf{r}, \mathbf{r}_a) w_n^{(k)}(l_k) dl_k = 0,
\]

\( n = 1, 2, \ldots, N; \ k = 1, 2, \ldots, P \),

thus obtaining a block-matrix linear system:

\[
[\mathbf{L}] \mathbf{b} = \mathbf{c}.
\]

The matrix \( [\mathbf{L}] \) is readily shown to be symmetrical, positive definite and hence non-singular. The components of \( \mathbf{b}, \mathbf{c} \) and \( \mathbf{L} \) are explicitly given by (5),

\[
\epsilon^{(k)}_i = -\int_{\sigma_k} w^{(k)}_i(l_k) g(l_k, \mathbf{r}_a) dl_k,
\]
\[ i = 1, 2, \ldots, N; \quad k = 1, 2, \ldots, P, \]  
and:
\[ [L_{M}^{(p,q)}]_{ij} = \int_{\sigma_{p}} \int_{\sigma_{q}} g(l_{p}, l_{q}) w_{i}^{(p)}(l_{p}) w_{j}^{(q)}(l_{q}) \, dl_{p} \, dl_{q}, \]
where the upper indexes identify the block sub-matrix, and the lower ones the element in each sub-matrix.

Using eq.s (2)-(5), once (7) has been solved, the Laslett coefficients can be computed without resorting to numerical differentiation. This makes the proposed method definitely more accurate than both finite-differences and finite-elements.

3 IMPLEMENTATION AND COMPUTATIONAL BUDGET

Fast and accurate numerical solution of (7) follows from a skillful choice of the RDGF representation in (4) and the basis functions in (5).

A rapidly converging series expansion of the RDGF [5], which explicitly contains the (logarithmic) singular term is \(^3\):
\[ g_{n}(\mathbf{r}, \mathbf{\ell}) = -\sum_{m=-\infty}^{\infty} \log \frac{T_{m}^{10}(\mathbf{r}, \mathbf{\ell}) T_{m}^{01}(\mathbf{r}, \mathbf{\ell})}{T_{m}^{10}(\mathbf{0}, \mathbf{0}) T_{m}^{00}(\mathbf{0}, \mathbf{0})}. \] (10)

where:
\[ T_{m}^{pq}(\mathbf{r}, \mathbf{\ell}) = 1 + \exp \left[ -2 \left| y - (-)^{p} y_{b} + 2 b m \frac{\pi}{a} \right| + 2 \exp \left[ y - (-)^{p} y_{b} + 2 b m \frac{\pi}{a} \cos \frac{\pi}{a} (x - (-)^{q} x_{b}) \right] \right], \] (11)
a, b being the rectangle side lengths.

A convenient set of (partially overlapping) piece-wise parabolic subdomain basis functions, can be defined in terms of the local angles \( \phi \) (we drop the suffix \( k \) for simplicity) as follows:
\[ w_{i}(\phi) = \frac{\Delta \phi^{2} - (\phi - \phi_{i})^{2}}{\Delta \phi^{2}}, \]
\[ \phi_{i} - \Delta \phi (1 - \delta_{i1}) \leq \phi < \phi_{i} + \Delta \phi (1 - \delta_{iN}), \]
\[ i = 1, 2, \ldots, N, \] (12)
where \( \Delta \phi \) is the angular discretization step (assumed the same for all arcs), \( \phi \) is related to the local arc-length \( l \) by \( l = R \phi, \) \( R \) being the local curvature radius, and \( \delta_{iN} \) is the Kronecker symbol\(^4\). The relevant local coordinate systems are sketched in Fig. 1. Note that: i) the choice of sub-domain basis functions, rather than full-domain ones, results into fewer singular integrals in \([L]^{ij}\); ii) no polygonal approximation of the arcs is implied, resulting into fewer functions being needed for a given accuracy.

Letting \( P \) the number of arcs in the rounded portion of \( \partial S_{0}, \) the system (7) has rank \( NP. \) Computing the matrix elements requires evaluating up to \( PN(\text{PN} - 1)/2 \) double-integrals\(^5\). These latter can be either evaluated numerically using standard routines appropriate for regular [7] and singular integrands [8], or analytically [4]. Matrix inversion for solving (7) is not the most demanding task, in view of the typically small (\( NP \approx 20 \)) \( L \) matrix size. In all numerical simulations below we truncated (10) at \( |m| \leq 3 \) and took \( \Delta \phi = \pi/10, \) corresponding to a matrix size \( NP = 20. \)

4 NUMERICAL RESULTS AND CONCLUSIONS

The circular pipe, for which the tune-shifts are known exactly, is the hardest conceivable test case for the proposed method (largest departure from rectangular geometry). It is seen from Fig. 2 that the obtained accuracy is very good.

Our method was subsequently applied [4] to a number of different proposed geometries relevant to LHC [3].

As an example the contour-level plots for the incoherent tune-charts of a stadium-shaped pipe, sketched in Fig. 3, are shown in Fig.s 4-6.

As a conclusion, we found that the above hybrid approach is comparatively faster and more accurate than available finite-element and/or finite-difference techniques.

5 REFERENCES


\(^{3}\)It is easily recognized that the (logarithmic) singularity of \( g_{R} \) appears in the \( T_{0}^{10} \) term.

\(^{4}\)For \( i = 1, N, \) eq. (12) yields the correct behaviour at the points where the circular arcs join the straight portions of \( \partial S_{0}, \) where \( \rho_{n} \) can be different from zero, but its derivative should vanish.

\(^{5}\)Due to geometrical (specular) symmetries, the effective number of elements to compute is usually smaller.
Fig. 1 - Local coordinate system relevant to eq. (12).

Fig. 2 - Circular pipe. Errors on Laslett coefficients vs. scaled radial distance, $p = \frac{1}{a} \left( \frac{x-a}{2} \right)^2 + \left( \frac{y-a}{2} \right)^2$.

Fig. 3 - Stadium-shaped pipe (a=1, b=0.7).

Fig. 4 - Stadium-shaped pipe. Incoherent Laslett coefficients (both normal modes).

Fig. 5 - Stadium-shaped pipe. Coherent Laslett coefficient (1st normal mode).

Fig. 6 - Stadium-shaped pipe. Coherent Laslett coefficient (2nd normal mode).