AND THIRD-ORDER

ACHROMAT DESIGNS

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I. INTRODUCTION

An achromat is a transport system that carries a beam without distorting its transverse phase space distribution. In this study, we apply the Lie algebraic technique [1-6] to a repetitive FODO array to make it either a second-order or a third-order achromat. (Achromats based on reflection symmetries [7,8] are not studied here.) We will consider third-order achromats whose unit FODO cell layout is shown in Fig. 1. The second-order achromat layout is the same except the octupoles are absent.

For the second-order achromats, correction terms (due to the finite bending of the dipoles) to the well-known formulae for the sxtupole strengths are derived. For the third-order achromats, analytic expressions for the five octupole strengths are given. The quadrupole, sxtupole and octupole magnets are assumed to be thin-lens elements. The dipole magnets are assumed to be sector magnets filling the drift spaces. More details of the analysis have been reported elsewhere.[9] We thank Y. Yan, H. Ye, J. Irwin and A. Dratt for their help.

II. ANALYSIS

We first calculate the Lie maps of each of the magnet elements. The map for a magnet element of length $L$ is given by $e^{-L:H}$, where $H$ is the Hamiltonian of the element. For a particle with $\delta = \Delta P/P_0$, we use (we ignore the path-length dynamics)

thin quadrupole : $H_1 = \frac{1}{2F_k}(x^2 - y^2)(1 - \delta + \delta^2)$

thin sxtupole : $H_2 = \frac{S_k}{3}(x^3 - 3xy^2)(1 - \delta)$

thin octupole : $H_3 = \frac{O_k}{4}(x^4 - 6x^2y^2 + y^4)$

sector dipole : $H = \frac{P^2}{2} + P_y^2 + \frac{x^2}{R} + \frac{x(2P_x^2 + 2P_y^2)}{2R} - \frac{x^2\delta}{R} + \frac{x\delta^2}{8} - \frac{x\delta^3}{2R^2} + \frac{x\delta^4}{8} - \frac{x\delta^5}{2R^3} + \frac{x\delta^6}{4R^2}$

(1)

where $R$ is the bending radius; $F_k$ is the focal length of the $k$-th quadrupole; $S_k$ and $O_k$ are the $k$-th integrated sxtupole and octupole strengths. Fringe fields are ignored.

Given the Hamiltonian $H$ of an element, we factorize the element map as

$$e^{-L:H} = e^{H_2 + H_3 + H_4 + \cdots} = e^{H_2}e^{H_3}e^{H_4}e^{O(X^3)};$$

(2)

where $H_2$ and $H_3$ are polynomials of order 2 in the variables $X = (x, P_x, y, P_y, \delta)$. We performed this factorization [3,5] and obtained

thin quadrupole :

$$f_3 = \frac{1}{2F_k}(x^2 - y^2)\delta, \quad f_4 = -\frac{1}{2F_k}(x^2 - y^2)\delta^2.$$ 

thin sxtupole :

$$f_3 = -\frac{S_k}{3}(x^3 - 3xy^2), \quad f_4 = \frac{S_k}{3}(x^3 - 3xy^2)\delta.$$ 

thin octupole :

$$f_3 = 0, \quad f_4 = -\frac{O_k}{4}(x^4 - 6x^2y^2 + y^4).$$

sector dipole :

$$f_3 = -\frac{1}{6R^2}\sin^3 \frac{L}{R} x^3 - \frac{1}{4R}\sin \frac{L}{R} \sin \frac{2L}{R} x^2 P_x - \frac{1}{6}(1 - \cos^3 \frac{L}{R}) P_x^3 x$$

$$-\frac{1}{2}\sin \frac{L}{2R} x P_y \sin^2 \frac{L}{2R} \sin \frac{L}{R} (cos \frac{L}{R} + \sin \frac{L}{R})$$

$$+ R\sin^2 \frac{L}{2R} P_x P_y - 2\sin^2 \frac{L}{2R} \sin \frac{L}{R} x P_x \delta$$

$$- \frac{R}{2}\sin^2 \frac{L}{2R} \sin \frac{L}{2R} P_x^2 \delta - \frac{1}{2}(L - R\sin \frac{L}{R}) P_y^3 \delta$$

$$- \frac{x^5}{2} \sin^3 \frac{L}{2R} \sin \frac{L}{R} x P_x \delta$$ 

$$+ \frac{1}{12}(-6L + 2R\sin^2 \frac{L}{R} + 3R\sin \frac{L}{R})\delta^3$$

$$f_4 = \left[-\frac{x^2}{8R^2}\sin^3 \frac{L}{R} - \frac{xP_x}{8R}\sin \frac{L}{R} \sin \frac{2L}{R} P_x^2 \delta$$

$$- \frac{R}{8}\cos \frac{L}{R} \sin \frac{L}{R} P_y^2 - \frac{8}{8}\sin \frac{L}{R} P_y^2 \right] (P_x^2 + P_y^2)$$

$$+ \frac{x^5}{12R^2}\sin^3 \frac{L}{R} L + (1 + \cos \frac{L}{R}) \sin \frac{L}{R} \sin \frac{L}{R} x P_x^2 \delta$$

$$+ \frac{R}{12}(3 + \cos \frac{L}{R}) \sin \frac{L}{R} \sin \frac{L}{R} x P_x^2 \delta$$

$$+ \frac{xP_x^2}{4}\sin \frac{2L}{R} + \frac{R}{4}(3 + \cos \frac{L}{R}) \sin \frac{L}{R} x P_x^2 \delta$$

$$- \frac{1}{4R}\sin^3 \frac{L}{R} L + \sin \frac{2L}{R} x P_y \sin^2 \frac{L}{R} \sin \frac{L}{R} x P_y \delta^3$$

$$- \frac{R}{4}(1 + \cos \frac{L}{R}) \sin^2 \frac{L}{R} \sin \frac{L}{R} P_y^2 \delta^3$$

$$+ \frac{x^5}{2}\sin^3 \frac{L}{2R} P_y + (\cos \frac{L}{R} + \frac{1}{4}\sin^2 \frac{L}{R}) \frac{x\delta}{2} \sin \frac{L}{R} \delta^2$$

$$- \frac{R}{2}\sin^3 \frac{L}{R} P_x \delta^3 + \frac{\delta^4}{12}(6L - R\sin^3 \frac{L}{R} - 3R\sin \frac{L}{R})\delta^3 \right)$$

(3)

Having factorized the maps of all magnets, the total map $M_{cell}$ of a cell is obtained by multiplying and concatenating the maps of the component elements:[3,9]
\[ M_{\text{cell}} = \prod_{i=1}^{N} (e^{iP_i} - e^{-iP_i}) = e^{i\Pi} - e^{-i\Pi} \cdot (1 - e^{i\Pi}) \]  

(4)

where

\[ R = e^{i\Pi} = \prod_{i=1}^{N} e^{iP_i}, \hspace{1cm} h_3 = \prod_{i=1}^{N} \tilde{P}_3 \]

\[ h_4 = \sum_{i=1}^{N} \tilde{P}_4 + \frac{1}{2} \sum_{j=1}^{N} \left[ \tilde{P}_3 \tilde{P}_3 \right] \]  

(5)

In Eq.(5), \( \tilde{P} \) means \( \tilde{P}(X) = f'((RX)_{\rightarrow X}) \) with \( RX_{\rightarrow X} \) the linear map from the last element to the \( i \)-th element. The map of the \( N \)-cell achromat is \( M = M_{\text{cell}}^N \). The number of cells \( N \) is so that \( \mu_{x,y} \) (the total phase advances in \( x \) and \( y \)) are both multiples of \( 2\pi \), but avoiding resonances.

We now make a canonical coordinate transformation from 

\[ (x, P_x, y, P_y) \rightarrow (\phi_x, A_x, \phi_y, A_y) \]  

where \( x = \sqrt{2A_x} \cos \phi_x + \eta \delta \), and similarly for \( y \) and \( P_y \) without the \( \eta \) and \( \eta' \) terms, where \( \beta_{x,y}, \alpha_{x,y}, \eta, \eta' \) are the Courant-Snyder and the dispersion functions.[10] The linear map generator \( h_3 \) becomes \( h_3 = -\mu_A A_x - \mu_y A_y - \frac{1}{2} \tilde{\alpha}_c \delta^2 \) where and \( \tilde{\alpha}_c \) is the momentum compaction factor. We then decompose \( h_3 \) in terms of the eigenvalues of \( h_3 \): as[5]

\[ h_3 = \sum_{\alpha b c d, \varepsilon} C_{\alpha b c d, \varepsilon} \epsilon \]  

(6)

To reduce a nonlinear map to its normal form, it can be shown[11] that (in the absence of resonances)[2] all the non-secular terms can be transformed away via a symplectic similarity transformation leaving only terms with \( \alpha = b = d = c, \) i.e., terms depending on \( A_x, A_y \) and \( \delta \) only. In particular, we have

\[ h_3 = \sum_{\alpha b c d, \varepsilon} C_{\alpha b c d, \varepsilon} \epsilon \]  

(7)

III. SECOND-ORDER ACHROMATS

For a second-order achromat, we follow Eqs.(6-7) and find the normal form of the unit cell is given by \( h_3 \) of Eq.(7) where

\[ C_{1100,1}^3 = \sum_{k=1,2} 2F_k \left( -\lambda_k \eta(k) \right) \beta_x(k) + w_x \]

\[ C_{0011,1}^3 = -\sum_{k=1,2} 2F_k \left( -\lambda_k \eta(k) \right) \beta_y(k) + w_y \]  

(8)

and

\[ w_x = \sum_{k=1,2} 2 \sin \left( \frac{L}{R} \right) \left[ \frac{\beta_x(k)}{\cos \frac{L}{R}} \right] + \frac{\eta(k)}{\sin \frac{L}{R}} - \eta(k) \cos \frac{L}{R} + \frac{\eta(k)}{\sin \frac{L}{R}} \cos \frac{L}{R} \]  

(9)

The lattice functions are evaluated at the two quadrupoles in Eq.(8) and at the ends of the two dipoles in Eq.(9). In the limit of weak bending with \( \epsilon_1 = \frac{L}{R} \ll 1 \), we have

\[ w_x \approx \epsilon_1 \sum_{\alpha} \alpha_x \sin \eta_x(s) + \frac{1}{4} \gamma_x(s) (3L \eta_y(s) - 2 \eta(s)) \]

(10)

To form a second-order achromat, we set the two C-coefficients to zero, and obtain

\[ S_1 = \frac{1}{2 \eta(1) F_1} \eta \left( \beta_x(1) \right) \beta_y(2) - \beta_x(2) \beta_y(1) \]

(11)

The first terms usually dominate and give the well-known results. The correction terms with \( w_x \) and \( w_y \) are normally but not always small.

IV. THIRD-ORDER ACHROMATS

We also studied the case of a third-order achromat. An algebraic program using Mathematica was developed to do the analysis. Here we only report our results. The normal form of the third-order generator for a unit cell is given by Eq.(9) with

\[ C_{2000,0}^3 = -\frac{3}{8} \sum_{k=1}^5 \beta_x(k) \beta_y(k) O_k + w_{xx} \]

\[ C_{1111,0}^3 = \frac{3}{2} \sum_{k=1}^5 \beta_x(k) \beta_y(k) O_k + w_{xy} \]

\[ C_{0022,0}^3 = -\frac{3}{8} \sum_{k=1}^5 \beta_y(k) \beta_x(k) O_k + w_{yy} \]

\[ C_{1100,2}^4 = -\frac{3}{2} \sum_{k=1}^5 \beta_x(k) \eta(k) O_k + w_{xd} \]

\[ C_{0011,2}^4 = \frac{3}{2} \sum_{k=1}^5 \beta_y(k) \eta(k) O_k + w_{yd} \]  

(12)

and (when \( \epsilon_1 = \frac{L}{R} \ll 1 \))

\[ w_{xx} \approx \csc^2 \frac{3 \mu_x}{2} (2 + 3 \cos \frac{\mu_x}{2}) \sum_{s} S_s \beta_x(s) \eta_x(s) \]

(13)

\[ + \frac{1}{2} \csc^2 \frac{3 \mu_x}{2} (2 + 3 \cos \frac{\mu_x}{2}) \sum_{s} S_s^2 \beta_x(s) \eta_x(s) \]  

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Exact expressions of the $w$-coefficients are too lengthy to be included here.

The required octupole strengths are such that the five $C$-coefficients in Eq.(12) are equal to zero. For the case when two of the octupoles are located next to the two sextupoles and the other three are at the $1 \over 3$, $2 \over 3$, and the $\frac{1}{2}$ locations of the two bending magnets, we find

$$O_1 \approx \frac{a + b}{6f^3D}, \quad O_2 \approx \frac{81(c + d)}{2fD}, \quad O_3 \approx \frac{81(c - d)}{2fD}$$

$$O_4 \approx \frac{a - b}{6f^3D}, \quad O_5 \approx \frac{128e}{3(2f^3 - 1)D}$$

$$a = 2f(1360 - 22846f^2 + 74476f^4 + 695809f^6 - 1438146f^{8} + 1200096f^{10} - 326592f^{12})$$

$$b = -352 - 3360f^2 + 233290f^4 - 1070910f^6 + 1917603f^{8} - 1364850f^{10} + 361584f^{12}$$

$$c = 6f(-42 + 1076f^2 - 7409f^4 + 16306f^6 - 14368f^8 + 4032f^{10})$$

$$d = 8 - 394f^2 + 5322f^4 - 16907f^6 + 14866f^{8} - 4464f^{10}$$

$$e = -368 + 10536f^2 - 92342f^4 + 307222f^6 - 470547f^8 + 330642f^{10} - 81648f^{12}$$

$$D = (4f^2 - 1)^2(3f^2 - 4)(10 - 173f^2 - 261f^4 + 324f^6) L^3 \epsilon_1^4$$

We have defined the dimensionless parameter $f = \frac{2\mu_0}{r}$ and have assumed that $\epsilon_1 = \frac{L}{r} \ll 1$ and $|S_x + S_y| \ll 1$.

References