THE PHYSICAL MECHANISM OF ULTRA-FAST AUTOMATIC COOLING FOR BEAMS IN THE SIX-DIMENSIONAL EMITTANCE SPACE

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Abstract
This paper is to illustrate that the four-dimensional emittance is automatically shrinked in the particular static magnetic field, while the vertical emittance is still a constant. It is the physical mechanism that the transverse velocity of any particles is not only decreased but also makes the longitudinal velocity approximate to the velocity of the equilibrium particle.

1. INTRODUCTION
The ultra-fast automatic cooling for beams (U-FACB) was found out [1]. In the work followed [2], the damping phase diagrams in the emittance space were obtained. Such show that UFACB of the emittance space in the transverse direction is true indeed. In this paper the result of UFACB in six-dimesional emittance space will be given, so we are sure that the phase density of the emittance space is increased indeed. By means of the simply and explicitly physical picture of the transfer between both of the transverse and longitudinal energy, the mechanism of UFACB is illustrated. The transverse cooling is due to the energy transfering to the longitudinal direction, which also results in the longitudinal cooling, so the six-dimesional cooling rate is much faster than the transverse cooling rate.

2. THE TRANS FORMATION OF THE PHASE VOLUME
The relativistic Hamiltonian for a charged particle moving under the influence of an electromagnetic field [3] is

\[ H = \frac{1}{2} (1 + k \cdot x)^2 (p - eA)^2 + (p_x - eA_x)^2 + (p_z - eA_z)^2 + m^2 c^2 + eV \]  

(2.1)

Since the particle velocity \( v = \text{const} \), let us introduce the relative velocity

\[ \frac{d}{dt} = v \cdot \frac{d}{dt} \]

\[ (1 + k \cdot x)^2 \cdot s^2 + x^2 + y^2 = 1 \]  

(2.2)

The corresponding canonical momentum can be represented by [3]

\[ p_x = \frac{e}{m} (1 + k \cdot x) \]

\[ = m \gamma \nu (1 + k \cdot x)^2 \cdot s + eA_x \]

\[ p_y = m \gamma \nu y + eA_y \]

in the uniform field, let

\[ \frac{e}{m \gamma \nu} B = k \]  

(2.4)

the corresponding vector potentials are

\[ A_x = \vec{A} \cdot \vec{s} \cdot (1 + k \cdot x) \]

\[ = - \frac{1}{2} \frac{p_x}{c} (1 + k \cdot x)^2 \]

\[ A_y = A_z = 0 \]

the eq. (2.3) becomes

\[ H = \left( \frac{1}{(1 + k \cdot x)^2} [p_x + \frac{1}{2} p_z (1 + k \cdot x)^2] \right)^{1/2} \]

(2.6)

Let us consider the four-dimensional emittance on the horizontal direction. In the canonical phase space, we have

\[ du = dx \cdot dp_x \cdot ds \cdot dp_z \]

(2.7)

in the emittance space, there is

\[ dv = dx \cdot ds \cdot ds \cdot ds \]

(2.8)

they are related to each other by Jacobian determinant \( J \);

\[ du = J dv \]

(2.9)

performing a derivitive to eq(2.5), we know,

\[ J = (m \gamma \nu)^2 (1 + k \cdot x)^2 \]

(2.10)

then

\[ du_x = (m \gamma \nu)^2 (1 + k \cdot x)^2 dv_x \]

(2.11)

\[ du_z = (m \gamma \nu)^2 (1 + k \cdot x)^2 dv_x \]

Liouville's theorem tell us,

\[ du_k = du \]

(2.12)

therefore

\[ dv_x = \frac{(1 + k \cdot x)^2}{(1 + k \cdot x)^2} dv_x \]

(2.13)

Which shows that the emittance element is not a con-
stant. In other words, the necessary and sufficient condition for the six-dimensional emittance conservation cannot be obtained.

3. THE MOTION ON THE MEDIAN PLANE

In the case of the uniform field, the vertical motion is equivalent to the motion in the free space, so that the vertical emittance is an invariant. In this way, we need only to discuss the motion in the median plane of the magnet.

Under the hard-edge approximation, as shown in Fig.1, when the initial conditions of the incident particles are \( x_i, x_i, s_i, s_i \) respectively, where dot means the derivative with respect to \( t \) over \( v \), the straight line equation of the particle’s orbit in free space is written as

\[
x - x_i = x_i' \cdot (s - s_i)
\]

where \( x' = \frac{dx}{ds} = \dot{x} / \dot{s} \). the \( z \)-axis equation is

\[
x = s \cdot \tan (\frac{\pi}{2} - \theta_a)
\]

\[
\tan (\theta_a) = \tan (\frac{\pi}{2} - s_i) \cdot \frac{x_i}{s_i} / \sin (\theta_a) + \frac{s_i}{s_i} \cdot \cos (\theta_a)
\]

performing the solution of the above two linear algebraic equations, we get the coordinate at the magnet entrance

\[
\Delta x = \frac{1}{1 - \sin^2 \theta_a}
\]

\[
\Delta s = \frac{1}{1 - \cos^2 \theta_a}
\]

where \( \Delta s \) stands for the travelling distance in the edge region.

Let us consider the transformation of the beam going through the uniform magnetic field region, as shown in Fig.1, the particle’s orbit in the uniform field is a circle. In the conditions given by the paper [1][2] the edge at the magnet exit is parallel to that at the entrance, and the center of the equilibrium orbit places on the exit edge. In that case, no one of the velocity is not parallel to each other at any place while they are parallel at the entrance.

By means of the similar relation, the increased quantity of the distance of travels in the magnet region can be written as [1, 2]

\[
\Delta S = \int \left[ \frac{1}{\sqrt{1 - t^2}} - \frac{1}{\sqrt{1 - t^2}} \right] \, dt
\]

we know

\[
x_i = -\sin (\theta_a)
\]

\[
\sin (\theta_a) = \left[ 1 - s_i \cdot \sin (\theta_a) + \frac{x_i}{s_i} \cdot \cos (\theta_a) + \frac{x_i}{s_i} \cdot \sin (\theta_a) \right]
\]

since the coordinates and the velocity are continuous, by means of the similar ways to the paper [1], the final results can be written as

\[
x = -\left[ \frac{s_i \cdot x_i - x_i \cdot s_i}{s_i \cdot \cos (\theta_a) + x_i \cdot \sin (\theta_a) + \sin (\theta_a)} \right] + \Delta x
\]

\[
x_i = \frac{\Delta x}{\Delta s}
\]

\[
\Delta s = \Delta s_i + \Delta s_m
\]

they are the precise solutions in the four dimensions. In the case of the two dimensions, those become the same as eq. (3.9) in the paper [1]

4. JACOBIAN TRANSFORMATION OF THE EMITTANCE SPACE IN THE FOUR DIMENSIONS

By means of differentiating the above precise formulas, the corresponding Jacobian determinant in the four dimensions can be obtained. Since \( x_i \) is only dependent on \( x_i \), and \( s_i \) is only dependent on \( s_i \), we know

\[
\begin{align*}
\frac{\partial x}{\partial x_i} & = \frac{\partial x_i}{\partial x_i} \\
\frac{\partial x}{\partial s_i} & = \frac{\partial s_i}{\partial s_i}
\end{align*}
\]

\[
\begin{bmatrix}
J_{11} & J_{12} & J_{13} & J_{14} \\
J_{21} & J_{22} & J_{23} & J_{24} \\
J_{31} & J_{32} & J_{33} & J_{34} \\
J_{41} & J_{42} & J_{43} & J_{44}
\end{bmatrix}
\]

\[
J = \begin{bmatrix}
J_{11} & J_{12} & J_{13} & J_{14} \\
J_{21} & J_{22} & J_{23} & J_{24} \\
J_{31} & J_{32} & J_{33} & J_{34} \\
J_{41} & J_{42} & J_{43} & J_{44}
\end{bmatrix}
\]

\[
J_{11} = \frac{\partial x}{\partial x_i} = \frac{s_i}{s_i} \cdot \cos (\theta_a) + x_i \cdot \sin (\theta_a)
\]

\[
J_{14} = \frac{\partial x}{\partial s_i} = \frac{x_i}{s_i} \cdot \sin (\theta_a) + x_i \cdot \cos (\theta_a)
\]

\[
J_{21} = \frac{\partial x}{\partial x_i} = \frac{s_i}{s_i} \cdot \cos (\theta_a) + x_i \cdot \sin (\theta_a)
\]

\[
J_{24} = \frac{\partial x}{\partial s_i} = \frac{x_i}{s_i} \cdot \sin (\theta_a) + x_i \cdot \cos (\theta_a)
\]

\[
J_{31} = \frac{\partial x}{\partial x_i} = \frac{s_i}{s_i} \cdot \cos (\theta_a) + x_i \cdot \sin (\theta_a)
\]

\[
J_{34} = \frac{\partial x}{\partial s_i} = \frac{x_i}{s_i} \cdot \sin (\theta_a) + x_i \cdot \cos (\theta_a)
\]

\[
J_{41} = \frac{\partial x}{\partial x_i} = \frac{s_i}{s_i} \cdot \cos (\theta_a) + x_i \cdot \sin (\theta_a)
\]

\[
J_{44} = \frac{\partial x}{\partial s_i} = \frac{x_i}{s_i} \cdot \sin (\theta_a) + x_i \cdot \cos (\theta_a)
\]
\[ J_{22} = \frac{a_{x_t}}{a_{x_t}} = -\left[ \cos(\theta_a) + \frac{x_i}{\sqrt{1-x_i^2}} \cdot \sin(\theta_a) \right] \]
\[ J_{31} = \frac{a_{x_t}}{a_{x_t}} = \sin(\theta_a) / \left[ \delta_t \cdot \cos(\theta_a) + x_t \cdot \sin(\theta_a) \right] \]
\[ J_{33} = \frac{a_{x_t}}{a_{x_t}} = \cos(\theta_a) / \left[ \delta_t \cdot \cos(\theta_a) + x_t \cdot \sin(\theta_a) \right] \]
\[ J_{44} = \frac{a_{x_t}}{a_{x_t}} = \frac{\tan(\delta_t)}{\tan(\delta_t)} \cdot \frac{1}{\sin(\theta_a) - x_t \cdot \sin(\delta_t) + x_t \cdot \cos(\theta_a) + \sin(\theta_a)} \]
\[
\tan(\delta_t) = \frac{\sin(\delta_t)}{1 - \sin^2(\delta_t)} \]

This result gives us inspirations as follows.

1. Jacobian \( J \) is less than unity when \( \delta_t \neq 0 \); the emittance in the four dimensions will be always shrunk. In other words, it is cooling indeed.

2. From the formula (4.1) we know, \( J_{11} \cdot J_{22} = 1 \), which signifies that the extended rate of the transverse coordinate \( x \) is just equal to the shrunk rate of the transverse velocity \( x_t \); so the transverse emittance in the six dimensions is an invariant, too. Therefore, the beam cooling is only taken place in the longitudinal direction.

3. Since the vertical emittance is an invariant, but one of the four degree daughter space is shrunk, it is sure that the emittance in the six dimensions is decreased.

**5. THE PHYSICAL MECHANISM**

The physical mechanism of the cooling can easily be seen. It is the basic characteristics of the motion that all of the transverse velocity not to equal zero are always decreased and transferred to the longitudinal. Firstly, \( J_{11} > 1 \), the dimension of the transverse amplitude for the parallel incident particles is increased; secondly, \( J_{22} < 1 \), the transverse velocity is decreased; and thirdly, \( J_{11} \cdot J_{22} = 1 \), the transverse emittance for the six dimensions is an invariant. In that case, the transverse velocity is continuously transferred to the longitudinal direction to make the longitudinal velocity continuously approximate to the equilibrium one. However, \( J_{33} \) is approximately equal to unity 1, but \( J_{44} < 1 \), the longitudinal dimension is approximately invariant, while the longitudinal emittance is obviously decreased. Since the longitudinal cooling comes from that the transverse velocity is transferred to the longitudinal direction, we can call it the transfer cooling.

In the paper[1][2], in the case of simplifying the problem to the four-dimesional one, transverse emittance is decreased, which is led by the longitudinal velocity transferred to the transverse. we know, \[ \frac{dx_t}{ds_t} = \frac{dx_t}{dr} \cdot \frac{dr}{ds_t} \]
\[ = x_t / s_t = -\tan(\delta_t) \]
and we have seen that the decreased rate of the transverse velocity \( x_t \) is equal to the increased rate of \( x' \), also we think about that the longitudinal velocity \( s_t \) is increased because \( x \) is transferred to the longitudinal direction, so the decreased rate of \( x' \) must be less than the decreased rate of \( x \) because of the increase of the denominator \( s \). Therefore, in this case, the transverse cooling comes from the longitudinal transfer.

The mechanism gives us an inspiration to find a way of improving the cooling rate. If we make the edge focusing angle \( \delta \) at the exit become minus, which plays a role in focusing for the transverse motion, the \( x_t \) at the exit is rather approximating to zero, so that the cooling rate must be improved, which will be tested latter.

**REFERENCES**

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