Effect of Space Charge Forces on Particle Tracking and Generation of High Order Maps*

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Abstract

TOPKARK is a beam optics program consisting of two Fortran codes developed in parallel: a 3-D high-order mapping code and a particle tracking code; both utilize a space charge model which treats the particle bunch as a uniformly-filled 3-D ellipsoid. The map code uses the differential algebra library DA [1] to generate an arbitrary-order Taylor map describing a given lattice, then the Lie algebra library LIELIB [2] is used to obtain the Dragt-Finn factorization [3] of the corresponding Lie polynomial. The Lie polynomial generated by TOPKARK without space charge has been successfully benchmarked through third order against MARYLIE 3.0 [4] and through fifth order against TLIE [5]. With space charge on, TOPKARK generates a linear map that agrees well with TRACE 3-D [6]. The tracking code uses a symplectic integration scheme [7] when space charge is off, and it includes a more general space charge model [8] which assumes only ellipsoidal symmetry of the spatial distribution.

I. GENERAL FEATURES OF THE CODE

TOPKARK has evolved from an earlier code, which was developed during a collaboration between Grumman, LBL and BNL [9]. The mapping version is a useful design tool, while the tracking version is a useful diagnostic which resolves any ambiguities regarding very high order effects that might be missed by Lie algebra or mapping codes and is also required for dynamic aperture studies.

A. Mapping Version

TOPKARK employs a fourth-order, adaptive-step-size, Runge-Kutta integration scheme [10] which provides good accuracy and reasonable computational speed. The differential algebra library DA [1] is used to generate a high-order Taylor map expansion of the dynamical variables about the design trajectory (in practice up to fifth order has been used) step by numerical step along the length of the lattice. This map is used to propagate the spatial moments of the (assumed) initial particle distribution from one integration step to the next.

At each integration step, a 3-D uniformly-filled ellipsoid is constructed according to the calculated spatial moments. The exact linear electric fields associated with this ellipsoid are calculated [11] and, in combination with any magnetic fields, are used to advance to the next step. At the end of the lattice, the final Taylor map is used to calculate the emittance and Twiss parameters of the final distribution. The Lie algebra library LIELIB [2] is used to obtain the Dragt-Finn [3] factorization of the Lie polynomial corresponding to this final Taylor map.

C. Features Common to both Versions

TOPKARK currently implements a number of “hard edge” or uniform-field magnet elements, including a dipole (i.e., a normal entry and exit sector bend) and quadrupole through duodecapole. Also available are “thin fringe” elements for dipole and quadrupole magnets. All of these elements, including the fringe fields, have been successfully benchmarked against MARYLIE 3.0 [4] through third order. The fringe

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field models, although calculated independently, were based on ideas developed previously by Forest [12].

TOPKARK also employs one extended-fringe magnet model. This is a line-dipole model for large-bore magnets constructed from a cylindrical array of magnetized rods, including quadrupole, octupole and duodecapole configurations [13]. TOPKARK was successfully benchmarked against TLIE [5] through fifth-order in a single test-case where these extended-fringe quadrupole and octupole models were used. New element types are easily added to the list above.

Both codes use MKS units, with all momenta normalized to the longitudinal design momentum $p_0$. We define the longitudinal variables $\delta\tau=\tau-\tau_0$ and $\delta\pi=(E_0-E)/\gamma_0p_0c$, and the magnetic rigidity $B_0=p_0/e$. For straight elements, the equations of motion are:

$$\frac{dx}{d\tau} = \frac{\beta_x}{\beta_z} \; ; \; \frac{dy}{d\tau} = \frac{\beta_y}{\beta_z} \; ; \; \frac{d\delta\tau}{d\tau} = \frac{1}{\beta_z} - \frac{1}{\beta_0}; \quad (1a)$$

$$\frac{dp_x}{d\tau} = \frac{1}{\beta_p}\left(\frac{\beta_y}{\beta_z} B_z - B_y + \frac{1}{\beta_z} E_x \text{ eff} \right); \quad (1b)$$

$$\frac{dp_y}{d\tau} = \frac{1}{\beta_p}\left(-\frac{\beta_x}{\beta_z} B_z + B_x + \frac{1}{\beta_z} E_y \text{ eff} \right); \quad (1c)$$

$$\frac{d\delta\pi}{d\tau} = -\frac{1}{\beta_p c}\left(-\frac{\beta_x}{\beta_z} E_x \text{ eff} + \frac{\beta_y}{\beta_z} E_y \text{ eff} + E_z \text{ eff} \right). \quad (1d)$$

The electric field components $E_x \text{ eff}$, etc. include the self-magnetic field of the particles and relativistic effects (see below). The corresponding equations of motion for bending elements have been given elsewhere [14].

II. SPACE CHARGE MODELS

We consider only models with ellipsoidal symmetry, meaning that the spatial density distribution has the form

$$p(x, y, z) = p_0 f(u), \quad (2a)$$

where the function $u(x, y, z)$ is defined by the equation

$$u^2(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, \quad (2b)$$

with $a^2=b^2=c^2$. Thus the one-parameter family of 3-D ellipsoids defined by Eq. (2b) are isodensity contours.

Such models yield electric fields of the following form (for all points within the distribution) [15], [8]:

$$E_{xB} = \frac{q_0 p_0}{2e_0} abc x \int_0^\infty ds \frac{f[u(x, y, z)]}{(a^2+s)^{3/2}(b^2+s)^{1/2}(c^2+s)^{1/2}}, \quad (3)$$

with analogous results for $E_{yB}$ and $E_{zB}$. These electric fields are calculated in the bunch frame, then relativistically transformed to the laboratory frame.

The length of the bunch as observed in the lab frame is shortened due to relativistic length contraction, so the lab frame distribution is first stretched out in the $z$-direction before calculating the fields: $\delta z_B = \gamma_0 \delta z_L$. This reduces the particle density: $\rho_B = \rho_L / \gamma_0$, so $E_{xB}, E_{yB}, E_{zB}$ are all reduced by a factor $\gamma_0$ from what one would naively calculate in the lab. However, the fact that the distribution is stretched out in $z$ effectively increases the value of $E_{xB}$ by $\gamma_0$ at the position of each particle, thus negating the decrease in $E_{xB}$ noted above. This stretching of the bunch also alters the geometry of the distribution, which affects the values of all three components of $E_B$ accordingly.

Particle velocities are neglected in the bunch frame, so the Lorentz transformations yield:

$$E_L^x = E_B^x; \quad E_L^y = E_B^y; \quad (4a)$$

$$B_L^x = B_B^x = 0; \quad B_L^y = 0; \quad (4b)$$

The Lorentz force equation is:

$$\frac{1}{\gamma} F_L = E_L + v \times B_L. \quad (4c)$$

Combining these results yields an effective electric field:

$$F_L \text{ eff} = F_B^x + F_B^y + F_B^z/\gamma_0. \quad (4d)$$

This is the quantity used to advance the particles. The longitudinal field is altered by geometric effects only, while the transverse fields are also reduced by a factor of $\gamma_0^2$.

A. Mapping Version

TOPKARK works with a Taylor-series expansion about the design trajectory, so the obvious question arises: How then does one propagate a particle distribution down the beamline? This has been explained in detail elsewhere [14], but essentially one calculates the second moments at a given point in the lattice by using second and higher moments of the initial distribution.

Of course, we must assume a convenient initial distribution function, and one that is consistent with our assumption of a uniformly-filled 3-D ellipsoid in space. Such a distribution has been found and implemented [14]. The projection of this distribution in the x-p-x plane has the form:

$$g(x,p_x) = \frac{3}{4\sqrt{10\pi} e_x} \left(1 - \frac{x^2}{5e_x}\right) \exp\left[-\frac{\beta_x}{2e_x}(p_x + \alpha_x x)^2\right], \quad (5)$$

for $x^2 \leq 5e_x \beta_x$ (otherwise, $g=0$). We are using RMS Twiss parameters, which means that $\langle x^2 \rangle = \varepsilon_x \beta_x$, $\langle p_x \rangle = -\varepsilon_x \alpha_x$, and $\langle p_x x \rangle = \varepsilon_x x$. The linear bunch frame electric fields can be found analytically in terms of complete elliptic integrals [11].

B. Tracking Version

Three distinct space charge models are being implemented in the tracking code. One assumes a uniformly-filled ellipsoid in space, for which $f(u)=1$. Another assumes a gaussian ellipsoid in space, for which $f(u)=\exp(-u^2/2)$. The third is a more
The algorithm used by TOPKARK cannot follow the development of any structure within a bunch (unless, in the case of the Garnett and Wangler scheme, such structures preserve ellipsoidal symmetry), such as might arise due to plasma waves or instabilities. Thus the transit time through a lattice should not be much longer than a plasma period [17].

IV. CONCLUSIONS

The mapping version of TOPKARK is a tested and reliable high-order optics code in which a TRACE3D-like space charge model has been successfully implemented. Implementation of three distinct space charge models—uniform, gaussian and more general—into the tracking version is currently under way.

Used in a complementary fashion, the mapping and tracking versions of TOPKARK will provide a unique tool for investigating basic physics issues associated with space charge. These codes will also serve as powerful design and diagnostic tools for high-brightness beam optics applications.

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VI. REFERENCES