Equivalent Equations and Incoherent Lifetime Calculated from $e^+e^-$ Beam-Beam Simulation

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Abstract

For given tunes $(\nu_x, \nu_y, \nu_s)$, the influence of different terms $x^k y^l s^m$ in the nonlinear beam-beam forces is different, so only some of them are important as sources of regular (nonstochastic) effects. As for irregular effects, it seems that in $e^+e^-$ colliders the simultaneous action of both quantum fluctuations and three-dimensional nonlinearities creates effectively an additional stochastic force. The developed program of beam-beam simulation including Fourier analysis of different moments $M^{kln}(t) = x^k(t)y^l(t)s^m(t)$ permits us to identify both regular and stochastic effective forces in the equivalent equations of the particles' motion. These differential equations can be used, for example, for the calculation of the particles' lifetime.

I. INTRODUCTION

The idea of extracting equivalent differential equations for particle coordinates $x$, $y$, and $s$ from incoherent spectrum densities calculated in the course of beam-beam simulations has been described in detail earlier [1,2]. The scheme of calculating the equivalent equations is the following.

The individual spectrum of a single particle #j is:

$$ m^{kln}_j(\nu) = \frac{1}{n_{\text{max}}} \sum_{n} \nu^k_j y^l_j s^m_j \exp(-2\pi i \nu n) $$

$\Delta \nu = 1/n_{\text{max}}$; $n$ = the number of revolutions.

Incoherent "Schottky noise":

$$ | F^{kln}_{\text{incoh}}(\nu) |^2 = \frac{1}{N} \sum_j | m^{kln}_j(\nu) |^2 $$

$N$ = the number of particles. $x$, $y$, $s$ are taken in the IP (interaction point).

An example of the effective equation for the particles’ vertical movement (as explained in [2]):

$$ \ddot{y} + \gamma \dot{y} + \omega^2(x, y) y = ay + b(x^2 - \langle x^2 \rangle) y + (c_o + c_B) \sum_{\nu_k < \nu} \delta(t - t_k) $$

(we can use the term $a_1 \dot{y}$ instead of $ay$). Analogous equations can be written for $x$ and $s$ oscillations.

II. DIFFUSION

$c_o$ corresponds to the radiation fluctuations without BB interactions.

$c_B$ describes the result of the combined effect of the radiation fluctuations and stochastic nonlinear diffusion.

For $\sigma^2 = \beta c$ we get

$$ \frac{d\sigma^2}{dt} = -2\gamma \sigma^2 + \text{const} \cdot (c_o^2 + c_B^2) \approx 0 $$

$$ \sigma^2 = \sigma_o^2 + \sigma_B^2 = \beta(\epsilon_o + \Delta \epsilon_B) $$

$$ \frac{\sigma_B^2}{\sigma_o^2} = \frac{c_B^2}{c_o^2} \approx \frac{\Delta \epsilon_B}{\epsilon_o}; \quad c_B \approx c_o \sqrt{\Delta \epsilon_B/\epsilon_o} $$

$$ \Delta \epsilon_B = \text{const} \cdot \int \Delta d\nu $$

(see Fig. 1).

$$ \epsilon_o = \text{const} \int | F^2(\nu) |^2 d\nu $$

(Using real, not logarithmic scales!)
IV. HORIZONTAL-VERTICAL COUPLING

\[ b \approx \frac{\omega r_{\nu} \nu_{H} \nu_{\nu}}{\sigma_{\nu}^{2} + \sigma_{H}^{2}} \sqrt{\frac{k_{D}}{H}} \]

V. FAST LOSS OF PARTICLES IN THE BEGINNING OF BB INTERACTION

The distribution before BB interaction:

\[ \phi_{o} = e^{-\frac{A^{2}}{2 \sigma_{o}^{2}}} \quad A^{2} = y^{2} + y_{0}^{2}/\omega_{o}^{2} \]

In the beginning of the BB, after \( \Delta t = \tau_{\text{rad}} \approx \frac{1}{\gamma} \)

\[ \phi_{B} = e^{-\frac{A^{2}}{2 \sigma_{B}^{2}}} \]

The technique previously used in [3] is used here. The number of particles \( n = \text{const} \int \phi dA^{2} \). \( \phi_{B} \) is the new distribution of the particles that survive after the beginning of BB plus \( \Delta t \). After that the slow loss begins.

The fast loss of particles (during \( \Delta t \approx \tau_{\text{rad}} \))

\[ \delta_y = \int_{\rho}^{\infty} \frac{dA^{2}}{2\sigma_{o}^{2}} \left( e^{-A^{2}/2\sigma_{o}^{2}} - e^{-A^{2}/2\sigma_{B}^{2}} \right) = 1 - \frac{\sigma_{o}^{2}}{\sigma_{B}^{2}} \]

The full fast loss is

\[ \delta = 1 - \frac{\sigma_{o}^{2}}{\sigma_{B}^{2}} \frac{\sigma_{o}^{2}}{\sigma_{o}^{2} + \sigma_{o}^{2}} \]

VI. LIFETIME OF THE BEAM CENTER DENSITY, \( n(0) \)

Here we give only an approximate estimation. We take into account only the diffusion mentioned in II. In this approximation, the lifetime \( \tau_{D} \) depends mainly (and very strongly) on \( \xi \), \( \tau_{D} = f(\xi) \), [3]

\[ \xi = \frac{A^{2}_{\text{(permit.)}}}{2\sigma^{2}} \]

\( A_{\text{(permit.)}} \) is the permitted amplitude (the distance to the physical border, or to the dynamic aperture).
σ is rather uncertain. For big vertical amplitudes, when $A_y \sim A_{\text{permit}}, z - y$ coupling is essential; so, $\sigma \sim \sigma_z$ (and not $\sigma_y$).

According to [3]

\[
\frac{n(0, t)}{n(0, 0)} \approx e^{(t - \xi)} \cdot e^{-2\gamma t e^{-\xi}}
\]

Here $t = 0$ means just after the fast loss. For $t = \tau_D = 4$ hours, $\gamma = 1/400T$, $\frac{n(0, t)}{n(0, 0)} = 1/e$ we get $\xi \sim 24$; $A_{\text{permit.}}/\sigma \sim 7$.

VII. REFERENCES

