Theoretical Minimum Emittance Lattice for an Electron Storage Ring

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Abstract

A theoretical minimum emittance lattice for an electron storage ring is derived, where the dispersion function at the entrance to the dipole is varied to minimize the \(< H >\) function in the dipole. We find that the achievable minimum emittance is about 1/3 that of the equivalent Chasman-Green lattice. The interesting aspect is that the optimal minimum beta value in the minimum emittance lattice is 4/3 times larger than that of the corresponding Chasman-Green Lattice. Therefore it may be easier to achieve the minimum emittance with this lattice. However, the theoretical minimum emittance lattice does not have a zero dispersion straight section.

1. Introduction

In recent years, electron storage rings have frequently been used as light sources for research in atomic, molecular, condensed matter and solid state physics, chemistry, cell biology, microbiology, and electronic technology processing etc. For many experiments, it is desirable to use high brightness light, which requires a high brightness electron beam. The synchrotron light emitted from a storage ring dipole spans vertically an rms angle of 1/\(\gamma\) around the beam trajectory at the point of emission, where \(\gamma\) is the Lorentz factor. Horizontally, the synchrotron light fans out an angle equal to the bending angle of the dipole magnet.

The synchrotron light spectrum is continuous with a critical energy of \(\hbar \nu_e = \frac{\gamma^2 - 1}{4 \gamma^2} \lambda_e m_e c^2\), where \(\lambda_e\) is the Compton wavelength of the electron.

The amplitudes of the betatron and synchrotron oscillations are determined by the equilibrium processes of the quantized emission of photons and the real acceleration fields used in compensating the energy loss of the synchrotron radiation\(^1\). The horizontal emittance is given by

\[
\epsilon_x = C_q \gamma \frac{\hbar H_{dipole}}{J_e \rho},
\]

where \(C_q = 3.84 \times 10^{-13}\) m, \(J_e \approx 1\) is the damping partition number and \(< H >\) is averaged over the dipole for the function,

\[
H = \frac{1}{\beta_x} \left( \eta_x^2 + (\alpha_x \eta_x + \beta_x \eta_x')^2 \right),
\]

where \(\alpha_x, \beta_x\) are the Courant-Snyder betatron amplitude functions; \(\eta_x, \eta_x'\) are the dispersion functions.

The design of low emittance optics is an important task in achieving high brightness electron bunches. There are several important ways to obtain small \(< H >\) using the Chasman-Green lattice\(^3\)–\(^5\) and/or FODO cells\(^6\). FODO cells, composed of interspacing quadrupole and dipole magnets units, are used most often in the collider design due to their simplicity and high packing factor. A Chasman-Green (CG) lattice is composed of cells with two bend achromats, which connect zero dispersion straight sections. Because of its unique properties, the CG lattice has been widely used in the design of synchrotron light sources\(^7\). An alternative method in achieving a small emittance would be using a wiggler or an undulator to increase the radiation loss at the zero dispersive straight section.

In this paper we shall study the theoretical minimum emittance obtainable in the storage ring without using wigglers or undulators. Minimum emittance can be contemplated through minimizing the \(< H >\) function with respect to the dispersion functions, \(\eta_x\) and \(\eta_x'\).

2. Minimum Emittance Chasman-Green

A half cell of the Minimum Emittance Chasman-Green (MECG) lattice is made of a single dipole with a set of quadrupoles on both sides such that (1) the dispersion function, \(\eta_x\), is zero on one side and finite on the other side; (2) \(\eta_x' = 0\) at the reflection symmetric point; and (3) the betatron amplitude function, \(\beta_x\), is shaped to have a minimum in the dipole region. The choice of the dispersion function gives rise to a zero dispersion straight section, which is beneficial for the rf cavities and insertion devices. With this choice of dispersion functions, we obtain\(^4\) the minimum betatron function location at \(s_{MECG}^* = \frac{3}{8} \ell_B\) with

\[
\beta_{MECG}^* = \frac{\sqrt{2}}{8 \sqrt{5}} \ell_B\]

where \(\ell_B\) is the length of the dipole, \(\rho\) and \(\theta\) are respectively the bending radius and the bending angle of the dipole. The final average value of the \(H\) function is

\[
<H >_{MECG} = \frac{1}{4 \sqrt{15}} \rho \theta^3\]

The corresponding \(H\) function at the ends of the dipole is given by \(H(0) = 0\) and \(H(\ell_B) = \rho \theta^3 \left( \frac{\sqrt{2}}{8 \sqrt{5}} + \frac{\sqrt{3}}{8 \sqrt{5}} \right)\).

3. Emittance Minimization Procedure

To obtain a minimum emittance, we have to minimize the average value of \(H(s)\) in the dipole region. Since the betatron amplitude function is shaped mainly by the quadrupoles, we shall assume certain desired properties for the betatron function in the dipole. Since the betatron amplitude functions outside the dipole region do not affect the emittance to first order, we can concentrate our discussion in the dipole region. There we shall solve for the dispersion function, which obeys the equation

\[
\eta_x'' + \frac{1}{\rho^2} \eta_x = \frac{1}{\rho},
\]

where the primes denote the derivative with respect to the longitudinal coordinate, \(s\), and \(\rho\) is the radius of the
curvature for the dipole magnet. Thus the general solution for the dispersion function is given by

\[ \eta = p(1 - a \cos \phi + b \sin \phi); \quad \eta' = a \sin \phi + b \cos \phi, \]

where \( \phi = s/p \). We have chosen \( s = 0 \) at the entrance of the dipole. The initial values of the dispersion function are given by \( \eta(s = 0) = 1 - a \) and \( \eta'(s = 0) = b \).

To minimize \( \langle H \rangle \) in the dipole, we assume \( \gamma = 1/\beta^* + (s - s^*)/\beta^* \)

which can be obtained by a proper arrangement of the quadrupoles. Other betatron functions are then given by:

\[ \alpha_x = -\frac{1}{\beta^*}(s - s^*)/\beta^*, \quad \gamma = 1/\beta^*. \]

Averaging the function \( H(s) \) of Eq.(2) in the dipole, we obtain

\[ < H > = \frac{\beta^2}{\beta^*} \left\{ \left( 1 - a^2 \right) \frac{1}{3} a(1-a)\theta^2 + \frac{1}{4}(1-a)b\theta^3 + \frac{1}{20}(a^2 - b^2)\theta^4 \right\} \]

\[ + \frac{1}{20} a^2 \theta^4 + \frac{1}{12} ab\theta^5 + \frac{1}{28} b^3 \theta^6 + [b^3 + ab\theta + \frac{1}{3}(a^2 - b^2)\theta^3] \]

\[ \frac{1}{4} ab\theta^3 + \frac{1}{20} b^3 \theta^4 + \frac{3}{10} ab\theta^5 - \frac{3}{12} b^3 \theta^6 \]

\[ + ab\theta + \frac{1}{4}(a^2 - b^2)\theta^2 - \frac{1}{4} ab\theta^3 + \frac{1}{20} b^3 \theta^4 \left( \frac{\beta^*}{\beta} \right)^2, \tag{5} \]

where \( \ell_B \) is the length of the dipole and \( \theta = \ell_B/\rho \) is the corresponding bending angle. \( < H > \) can be minimized with respect to \( s^* \). We obtain with \( s^* = s_m \) where \( s_m = \ell_B \left[ -b(1-a)\theta + a(1-a)\theta^2 + \frac{1}{3} a(1-a)\theta^2 + \frac{1}{28} b^3 \theta^6 - b\theta^2 \left( \frac{\beta^*}{\beta} \right)^2 \right] \) and \( B = b^3 + ab\theta + \frac{1}{3}(a^2 - b^2)\theta^2 - \frac{1}{4} ab\theta^3 + \frac{1}{20} b^3 \theta^4 \).

The resulting value of \( < H > \) is then given by

\[ < H > = \frac{\beta^2}{\beta^*} \left\{ A + B \left( \frac{\beta^*}{\beta} \right)^2 \right\}, \tag{6} \]

where the coefficients \( A \) and \( B \) can be obtained easily from Eq.(4), i.e. \( A = (1 - a^2 - \frac{1}{3} a(1-a)\theta^2 + \frac{1}{4}(1-a)b\theta^3 + \frac{1}{28} b^3 \theta^6 - b\theta^2 \left( \frac{\beta^*}{\beta} \right)^2 \) and \( B = b^3 + ab\theta + \frac{1}{3}(a^2 - b^2)\theta^2 - \frac{1}{4} ab\theta^3 + \frac{1}{20} b^3 \theta^4 \).

Varying \( \beta^* \), we obtain the minimum of \( < H > \) as

\[ < H > = 2B \cdot \beta^* = 2p\sqrt{AB} \tag{7} \]

with \( \beta^* = p \cdot \sqrt{\frac{s}{B}} \). Eq.(7) indicates that the achievable minimum emittance is proportional to the product of the coefficients \( A \) and \( B \). Therefore we will minimize the product of \( AB \) with respect to the dispersion functions in the dipole region. The Chasman-Green lattice corresponds to \( a = 1 \) and \( b = 0 \). In the following, we shall study some other cases.

3.1 Special case: \( b = \eta'(s = 0) = 0 \)

If we relax the constraint \( a = 1 \) but keep \( b = 0 \), we obtain a solution by minimizing the product of \( AB \) with respect to \( a \), i.e.

\[ a = \frac{180 - 15\theta^2 + \sqrt{3000 - 600\theta^2 - 135\theta^4}}{240 - 40\theta^2 + 3\theta^4} \approx \frac{1}{1 - \theta^2/12}, \]

It is interesting to note that the solution \( a = 1/(1 - \theta^2) \) corresponds to the condition \( s_m = \frac{1}{2} L_B \), i.e. the optimal location of the \( \beta^* \) is located at the center of the dipole. The optimal \( \beta^* \) value is \( \beta^* = \frac{2}{3} \cdot \frac{\sqrt{3}}{\pi} p\theta^3 \), which is 2/3 of the corresponding value of \( \beta^* \) in a MECG lattice. Because of the location of \( \beta^* \), the aperture requirement is about the same. Thus it is advantageous to obtain the minimum \( \beta^* \) at the center of the dipole. The maximum values of the betatron function on both side of the dipole will have the same magnitude. We will thus assume the approximate solution of \( a = 1/(1 - \theta^2) \) and obtain then \( < H > = \frac{\beta^2}{\beta^*} \left\{ \frac{1}{3} \theta^2 \theta^2 \right\} \left( \frac{\beta^*}{\beta} \right)^2. \tag{5} \)

The result can be interpreted easily. The minimum \( < H > \) is obtained from a slightly negative initial dispersion function, \( \eta(s = 0) = 1 - a = -\theta^2 \). The resulting emittance equals approximately 2/3 that of the corresponding MECG lattice. The difficulty in obtaining the minimum betatron amplitude function is the same as that with the Chasman-Green lattice. The \( H \)-function at the ends of the dipole is given by \( H(0) = \rho \theta^3 \left\{ \frac{1}{3} \theta^2 \theta^2 \right\} \left( \frac{\beta^*}{\beta} \right)^2 \) and \( H(\ell_B) = \rho \theta^3 \left\{ \frac{1}{3} \theta^2 \theta^2 \right\} \left( \frac{\beta^*}{\beta} \right)^2 \). Thus the dispersion function outside the dipole remains small.

3.2 General case:

If we also relax the condition \( \eta'(s = 0) = b = 0 \), the emittance can be minimized further. Constraining the minimum \( \beta^* \) at the location \( s_m = \frac{1}{2} L_B \), we obtain from Eq.(7)

\[ \theta(1 - \theta^2) a^2 - (-2b + \theta + b\theta^2 + \frac{1}{20} b\theta^4) a \]

\[ -b(2 + \theta - \frac{1}{3} \theta^2 + \frac{1}{12} b\theta^3 - \frac{1}{30} b\theta^4) = 0. \tag{8} \]

Table 1. Minimum emittance vs. initial dispersion functions with \( \theta = 2\pi/100 \) radians

<table>
<thead>
<tr>
<th>( b )</th>
<th>( a )</th>
<th>( \beta^* )</th>
<th>( \beta^*_{\min} )</th>
<th>( \beta^*_{\text{MCG}} )</th>
<th>( \epsilon_{\text{min}}/\epsilon_{\text{MCG}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>1.00932</td>
<td>0.6667</td>
<td>0.6671</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.01</td>
<td>1.00901</td>
<td>0.8620</td>
<td>0.5162</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.02</td>
<td>0.99976</td>
<td>0.9890</td>
<td>0.3942</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.03</td>
<td>0.99938</td>
<td>1.3316</td>
<td>0.3347</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.04</td>
<td>0.99907</td>
<td>1.2068</td>
<td>0.3687</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.05</td>
<td>0.99875</td>
<td>0.9328</td>
<td>0.4767</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1 lists the improvement factor, \( \epsilon_{\text{min}}/\epsilon_{\text{MCG}} \), for an accelerator composed of 100 half cells, i.e. \( \theta = 2\pi/100 \) for the storage ring. Note that the emittance is reduced when the constraints of \( a = 1 \) but \( b = 0 \) are relaxed. When the dipole bending angle \( \theta \) is varied, the optimal \( a, b \) values are also changed. We found that the optimal \( b \) value is about \(-\theta/2\). The resulting minimum emittance is about 1/3 of the corresponding Chasman-Green lattice. The above statement can be proved easily as following: Up to the lowest order in \( \theta \), Eq.(8) gives

\[ b = -\theta/2, \quad a = 1 - \theta^2/6. \tag{9} \]
where we obtain also \( r_m^* \approx \frac{1}{3} \epsilon_B \), \( A = \frac{1}{12} \theta^4 + O(\theta^8) \) and \( B = \frac{1}{12} \theta^5 + O(\theta^8) \). From Eq.(7), we obtain

\[
\beta_m^* = \frac{4}{3} \beta_{\text{MEG}}^*.
\]  

(10)

The \( H \)-function at the ends of the dipole are given by

\[
H(0) = \frac{1}{\sqrt{15}} \rho \theta^3; \quad H(\ell_B) = \frac{1}{\sqrt{15}} \left[ 1 + 2 \theta^2 + \frac{17}{12} \theta^4 \right] \rho \theta^3.
\]

Note that the values of the \( H \)-function at both ends of the dipole are small.

4. Beam Dynamics Properties

Finally, let us study the properties of the dispersion function outside the dipole region. The dispersion function outside the dipole region satisfies the same equation of motion as that of the horizontal coordinate. Thus the \( H(s) \) function is invariant. This means that \( \eta \) and \( \eta' \) are located on an invariant ellipse. Thus the only important quantities are the invariants at \( H(s = 0) \) and \( H(s = \ell_B) \). Maintaining a small value for these two quantities will guarantee a reasonable dispersion function in the quadrupole matching section.

For the maximum brilliance of the photon beam from an undulator located in the straight section at the entrance end of the dipole, one wants to minimize the beam width and not just the emittance. Let us discuss the general minimum emittance lattice discussed in section 3.2. The emittance is given by

\[
\epsilon = C_q \frac{J_{zp}}{\epsilon_B} \parallel H \parallel_{\text{dipole}} = \frac{1}{12 \sqrt{15}} C_q \frac{J_{zp}}{\epsilon_B} \rho \theta^3.
\]  

(11)

The corresponding dispersion emittance, which should be defined as

\[
\epsilon = \eta_2 \left( \eta_2 \delta^2 + 2 a_2 (\eta_2 \delta)(\eta'_2 \delta) + \beta_2 (\eta'_2 \delta)^2 \right) = H(0) \sqrt{\delta^2},
\]

where \( \delta^2 = (\frac{\epsilon}{B})^2 = C_q \frac{J_{zp}}{\epsilon_B} \) is the equilibrium energy spread in the beam. Thus substituting \( H(0) \) of section 3.2 into Eq.(12), we obtain then

\[
\epsilon_0 = \frac{1}{3 \sqrt{15}} C_q \frac{J_{zp}}{\epsilon_B} \rho \theta^3.
\]  

(12)

For a separated function lattice, \( J_F \approx 2, J_2 \approx 1 \) or \( J_F \approx 2 J_2 \). The total emittance for a bi-Gaussian distribution is given by

\[
\epsilon = \epsilon_B + \epsilon_0 = \frac{1}{4 \sqrt{15}} C_q \frac{J_{zp}}{\epsilon_B} \rho \theta^3 = \epsilon_{\text{MEG}}.
\]

(14)

Thus the decrease in the betatron emittance is taken up by the dispersion beam size. The brilliance of the photon beam (namely the size of the electron beam in the "dispersion free" straight section) is not affected by the dispersion introduced to minimize the betatron emittance. The total electron beam size in the straight section remains unchanged. Thus the minimization procedure does not impair the function of undulators. On the other hand, larger \( \beta^* \) may be helpful in the chromatic correction of the lattice.

5. Conclusion

In conclusion, we have relaxed the constraints of the lattice design to obtain a minimum emittance lattice for electron storage rings. We derived general properties of the minimum emittance lattice and compared them to that of the Chasman-Green lattice. We found that an emittance of about 1/3 of the equivalent Minimum Emittance Chasman-Green lattice can be obtained. Table 1 shows that optimal emittances are derived at \( \beta^* \) values larger than those of the Chasman-Green. The beam size in the straight section should remain the same as that of the MEG lattice. Thus the brilliance of the photon beam is not affected. Due to a smaller emittance, the photon brilliance should be greater in the dipole region.

Note however, the small \( \beta^* \) value remains to be an essential element in achieving a small emittance. The lattice would still be sensitive to errors. Thus careful studies are needed to evaluate the feasibility. Problems, such as chromaticity correction, sensitivity of the lattice perturbations, stopband widths, tunability, and stability arise in any lattice with small betatron amplitude functions. Careful studies of these problems are needed to understand the applicability of this minimum emittance concept. Possible retuning of the existing synchrotron radiation sources can be used to test the feasibility of the minimum emittance lattice.

Reference

5. see e.g. S. Tazzari, "Electron Storage Rings for the Production of Synchrotron Radiation", in CERN school proceedings, CERN 85-19, pp 566-585 (1985).