THE SPIN MOTION CALCULATION USING LIE METHOD
IN COLLIDER NONLINEAR MAGNETIC FIELD

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Abstract
Lie operator method of solving the spin motion equation in collider nonlinear fields is used. The matrix presentation of spin Lie transformation for particle passing through collider elements is obtained. The formulas for combined several spin turn transformations are calculated in vector, matrix and operator forms for zero, first and second powers component of dynamical variable vector. The expressions for frequency precession vector components in zero, first and second powers on orbit motion and first powers on spin motion are obtained. The computer codes algorithms for nonlinear spin motion calculation are discussed.

Solution of spin motion
As is known [1], the classical equation of spin motion in the collider is:
\[ \frac{dS}{ds} = [WS], \] (1)
where \( S \) is a spin vector, \( s \) is an azimuth and the precession frequency vector \( W \) is defined by BMT's equation [2]. The equation (1) is written in the frame \((e_x, e_z, T)\), fixed relative to the collider.

In the approach, which is based on using the technique of Lie operators, the vectors \( S \) and \( W \) in the equation (1) are considered as operators. Then for a particle with the orbital Hamiltonian \( H_{orb} \) and spin Hamiltonian \( WS \) one can find the solution of this equation (the semicolons ("::") emphasize the operator nature of the expression):
\[ S(s) = \exp(-\int_0^s (H_{orb} + WS) : : : dS) S(0). \] (2)

Here, as usual, the exponential operator is understood as a series:

\[ \exp(-\int F : : : dS) = \sum_{n=0}^{\infty} (-1)^n \frac{\int F : : : dS^n}{n!}, \]

Each term of this series is the differential operator of \( n \)-th power, which action on an arbitrary function \( f \) is defined with a help of Poisson brackets:
\[ F : : : f = (F, f) = \frac{df}{d\xi_1} \frac{df}{d\xi_2} - \frac{df}{d\xi_2} \frac{df}{d\xi_1}. \]

As is known, Poisson brackets are: \([q_1, q_2] = [p_1, p_2] = 0, q_1, p_k = -[p_1, q_k] - \delta_{1k}, \]
where \( q, p \) - are a conjugate dynamical variables pairs \((x, p_x), (z, p_z), (\sigma, p_\sigma), (J, \phi)\) and \( \delta_{1k} \) - is the Kroneker symbol. Since the spin motion equation can't be linearized in spin canonical variables action-angle \( J, \phi \), it is useful to introduce the set of noncanonical variables [3]:

\[ S_x = \sqrt{S^2 - J^2} \cos \phi, \quad S_y = \sqrt{S^2 - J^2} \sin \phi, \quad S_z = J, \]

where \( S^2 = \epsilon_x^2 + \epsilon_z^2 + \epsilon_\phi^2 \). For this set one can find:
\[ (S_1, S_2) = \epsilon_{ijk} S_k, \quad \epsilon_{ijk} \mbox{ is the three-dimensional completely antisymmetric tensor.} \]
Besides that for any \( i, j \) the following expression takes place:
\[ (\epsilon_i, \epsilon_j) = 0. \]

The operator, which is introduced in this manner is referred to as a Lie operator whereas the exponential series - Lie transformation.

For the total Hamiltonian \( H_{orb} + WS \), which does not depend on azimuth explicitly, one can find instead of (2):
\[ S(s) = \exp(-\int_0^s (H_{orb} + WS) : : : dS) S(0) = M S(0), \]
where \( M \) is a total exponential operator. According to the Hamilton equations, this operator satisfies an equation
\[ \frac{dM}{ds} = M : : : (H_{orb} + WS) : : : . \]
Let us present this operator as a product of three exponential operators [4,5]:
\[ M = M_2 M_1 M_0. \]

To this end expand the total Hamiltonian in a sum of homogeneous polynomials in powers of \( z \):

\[ H_{orb} + WS = H_2 + H_3 + H_4 + H_5 + H_6 + H_7 + \ldots, \]

where subscripts show powers of polynomials. It is important to find of \( M_2, M_1 \) and \( M_0 \), that operators \( H_2 \) and \( W_0 S \) do not change the power of dependence on \( z \) for any operands, but operators \( H_3 \) and \( W_1 S \) increase it by one. Similarly the operators \( H_4 \) and \( W_2 S \) increase it by two etc.

Using the Lie technique [6,7] of calculations \( M_2, M_1 \) and \( M_0 \) one can find, that
\[ M = M_2 M_1 M_0 = \exp(-:f_2:) \cdot \exp(-:f_1:) \cdot \exp(-:f_0:) = \]
\[ = \left( e^{-f_1} - :f_2: + \frac{f_1^2}{2} \right) M_0, \]
where \( :f_0: = :h_2 + W_0 S: = :S: H_2 + W_0 S : : : \)

operator \( M_0 \) is
\[ M_0 = \exp(-:f_0:) \cdot \exp(\phi W_0 S:) \cdot \exp(:h_2:) = S A \]
and
\[ :f_1: = :h_3 + W_1 S: = - \left( \int_0^s ds' M_0(s')(H_3 + W_1 S) \right), \]
\[ :f_2: = :h_4 + W_2 S: = - \left( \int_0^s ds' M_0(s')(H_4 + W_2 S) \right) - \frac{1}{2} \left( \int_0^s ds'\int_0^s ds'' M_0(s) M_0(s') \left( H_3 + W_1 S \right) \left( H_3 + W_1 S \right) \right) \].

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In this formulas $E$ is a unit operator, $S_0$ is a usual spin and orbital matrixes $S$ and $A$ of linear transformation and $[ , ]$ - the commutator of operators. Functions $h_1$ and $w^i s$ are "integrated" on azimuth polynomials with power equals $i$ on components of $Z$ for orbital hamiltonian $H_1$ and spin hamiltonian $H_2$.

For calculation one need to know the series of orbital Hamiltonian and spin precession frequency $w$ (sound from the BHT equation [7]), which are presented in supplement. The operators for different elements may be grouped in one operator [7]. Therefore spin dynamical characteristics may be investigated with using operators of one elements or group of elements (one or some period of collider in particularly).

### Polarization calculation

The degree of the equilibrium polarization is given by the Derbenev-Kondratenko formula [8]:

$$P_{eq} = \frac{a_+}{\alpha_+} = \int \frac{ds}{L} e_z (n - \frac{dn}{dy}) K^3,$$

$$a_+ = \int \frac{ds}{L} \left[ 1 - \frac{2}{9} (\frac{dn}{dy})^2 + \frac{11}{16} (\frac{dn}{dy})^2 \right] K^3,$$

where operator $M$ is determined in (3). Let us rewrite the $f_1$ and $f_2$ as coefficients of polynomials:

$$f_1 = h^2 pqr z_p z_q z_r \xi_1,$$
$$f_2 = h^2 pqr z_p z_q z_r \xi_2 = w^2 pr z_p z_q \xi_2.$$

For illustration let us find the results of action $f_1$ on $z$ and $s$. The Poisson brackets are:

$$\{ h, z \} = 3 h,$$
$$\{ h, s \} = 3 h.$$
\[
\begin{align*}
W_z &= -(B_{OZ} - K_x) - B_{OZ} \cdot (Y_o a^3 / 2 Y_o + \frac{1}{2} Y_o^2) - \\
&\quad \left[ \frac{1}{2} B_{OZ}^2 \cdot (Y_o - 1) + (B_{OZ} K_x + g) \cdot (1 + Y_o a) \right] \cdot x + \\
&\quad \left( B'_{OS} + q \right) \cdot \frac{1}{2} B_{OZ} \cdot (Y_o - 1) \cdot P_x + B_{OZ} \cdot P_o - \\
&\quad - \left( g K_x + \frac{1}{2} m_k \right) \cdot (Y_o a + 1) \cdot x^2 + \\
&\quad + (q K_x - g K_x - m_z) \cdot (Y_o a + 1) \cdot x \cdot z + B'_{OS} \cdot Y_o a \cdot x \cdot p_z + \\
&\quad (B_{OZ} K_x + g) \cdot P_o - \frac{1}{2} B_{OZ} (Y_o a + 1) \cdot P_x^2 + B_{OZ} \cdot P_a \cdot P_k + \\
&\quad + \left( \frac{1}{2} g K_x + \frac{1}{2} B_{OZ} \cdot B''_O \right) (Y_o a + 1) \cdot z^2 + B'_{OS} \cdot Y_o a \cdot z \cdot p_z - \\
&\quad - \left( \frac{1}{2} g B_{OZ}^2 \cdot (2 Y_o a + 1) \right) \cdot z^2 + \\
&\quad - \left( q + B'_{OS} \right) \cdot Y_o a \cdot z \cdot P_z - \frac{1}{2} B_{OZ} \cdot (2 Y_o a + 1) \cdot P_z^2 + \\
&\quad + B'_{UC} \cdot z \cdot P_o - B_{US} P_u^2.
\end{align*}
\]

In this expression \( y_o \) is the relativistic factor and \( a = 1.159 \ldots 10^{-3} \) is the dimensionless part of the electron anomalous magnetic momentum.

REFERENCES


