HIGH FREQUENCY DEPENDENCE OF THE COUPLING IMPEDANCE FOR A LARGE NUMBER OF OBSTACLES

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Introduction

We have recently derived an integral equation for the axial electric field at the pipe radius in the presence of an azimuthally symmetric cavity of arbitrary shape in a beam pipe of circular cross section. We have further shown that the local average of the coupling impedance over frequency decreases as \( k \) for high frequency, essentially independent of the cavity shape. In another paper, we extend the derivation to several cavities and obtain the high frequency behavior for a periodic cavity. In this case the real part of the impedance per cell is shown to vary as \( k^{1/2} \), in agreement with Heifen and Kheifets, and the imaginary part varies as \( k \), as required by causality.

In the present paper we analyze the case of \( N \) cavities and explore the high frequency behavior for large \( N \), in an effort to understand the transition to a periodic structure. Not unexpectedly, the result depends critically on which of the limits (\( k \to \infty \) or \( N \to \infty \)) is taken first.

Analysis

The starting point for the analysis is the integral equation obtained for the axial electric field at the beam pipe radius. Specifically we have

\[
\int_0^g dz' \mathbf{G}(z') \left[ \mathbf{K}_D(z - z') + \mathbf{K}_C(z', z) \right] = j \tag{2.1}
\]

where \( \mathbf{K}_D(z - z') \) and \( \mathbf{K}_C(z', z) \) are the modified "pipe" and cavity kernels, respectively. The solution of Eq. (2.1) with the kernels in Eqs. (2.4) and (2.5) then yields the "smoothed" high frequency limit for the impedance for a single obstacle:

\[
\mathbf{Z}_o = \mathbf{Z}_o Y(k) = \mathbf{F}_o(k), \quad \mathbf{F}_o(k) = \frac{(1 + j) \pi a n \sqrt{k a}}{2} \tag{2.6}
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The coupling between different cavities occurs through the pipe kernels, whereas the cavity kernels are diagonal. If we now use the high frequency kernels in Eqs. (2.4) and (2.5) for the diagonal terms and Eq. (2.3) for the pipe kernel in the coupling terms, it is clear that the only surviving contribution to the sum over \( m \) will be those for \( z_m' < z_n' \), that is \( m < n \). Specifically we obtain

\[
\sum_{m<n} \int_{z_m'}^{z_n'} dt' \mathbf{G}(t' - t) + \delta_{m,n} \mathbf{C}(z_m', z_n') = j, \tag{2.7}
\]

where \( z_m' \) and \( z_n' \) denote the variables \( z' \) and \( z \) within cavities \( m \) and \( n \), and \( \int_{z_m'}^{z_n'} dt' \) is over cavity \( m \). The coupling between different cavities occurs through the pipe kernels, whereas the cavity kernels are diagonal. If we now use the high frequency kernels in Eqs. (2.4) and (2.5) for the diagonal terms and Eq. (2.3) for the pipe kernel in the coupling terms, it is clear that the only surviving contribution to the sum over \( m \) will be those for \( z_m' < z_n' \), that is \( m < n \). Specifically we obtain

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where \( z' = nL + t' \), \( z_n = nl + t \), and where we assume that we have \( N \) identical cavities whose centers are spaced a distance \( L \) apart. We have also approximated \( z_n = z' \) by \( (n - m)L \) in the non-diagonal terms, corresponding to the assumption \( NL \gg g \). The impedance is then

\[
Z(k) = \frac{1}{z_0} \frac{1}{\pi \sqrt{ka}} \sum_{m=1}^{N} \int_{0}^{\infty} dt \frac{G_m(t)}{z_n} . \tag{2.9}
\]

Equation (2.8) can be simplified by writing

\[
G_n(t') = \frac{(1 - j) \sqrt{z}}{4\pi \sqrt{\sqrt{t'}}} y_n , \tag{2.10}
\]

leading to

\[
y_n + \frac{(1 - j) \sqrt{z}}{a \sqrt{\pi k}} \sum_{m=1}^{n-1} \sum_{m=1}^{N} y_m \exp \left( \frac{j(n - m)L_j^2}{2ka^2} \right) = 1 \tag{2.11}
\]

and

\[
Z(k) = \frac{(1 - j) \sqrt{z}}{z_0} \frac{1}{2\pi \sqrt{ka}} \sum_{n=1}^{N} y_n . \tag{2.12}
\]

Our task is to solve Eq. (2.11) for \( y_n \) and then use Eq. (2.12) to obtain the impedance. This can be facilitated by constructing the transform

\[
w(h) = \sum_{n=1}^{N} h^2 y_n ,
\]

in which case use of the convolution theorem leads to the solution

\[
w(h) = \left[ \frac{1}{1 + \frac{(1 - j) \sqrt{z}}{2\pi \sqrt{ka}} \rho(h)} \right]^{-1} , \tag{2.13}
\]

where

\[
\rho(h) = \sum_{n=1}^{N} \sum_{n=1}^{N} \frac{h^2}{2ka^2} \frac{1}{h} . \tag{2.14}
\]

The last form of Eq. (2.14) holds in the range

\[
ka^2 \gg L_j^2 .
\]

A simple approximation to \( \rho(h) \) in Eq. (2.12) for large \( N \) can be obtained by evaluating

\[
w[\exp(-1/N)] = \sum_{n=1}^{N} \frac{1}{N} e^{-n/N} , \tag{2.15}
\]

where the exponential cut-off simulates the sum from \( n = 1 \) to \( N \) in Eq. (2.12). For \( h = 1 - l/N \), we find

\[
\left(1 - \frac{1}{N} \right) \sum_{n=1}^{N} \left[ \frac{1}{\sqrt{\sqrt{ka^2} \frac{1}{\sqrt{ka}}} \frac{l}{\sqrt{ka}}} \left[ \frac{1}{N} \frac{l}{\sqrt{ka}} \frac{2ka^2}{2ka^2} \right] \right]^{-1} . \tag{2.16}
\]

Let us first consider the limit \( N \to \infty \). In this case we use \( \sum_{n=1}^{N} \frac{j2^2}{N} = 1/4 \) to evaluate the sum over \( s \), to obtain

\[
N Z_Y(k) \equiv \frac{(1 + j) \sqrt{\pi ka^2}}{\sqrt{\pi ka^2}} + \frac{\sqrt{ka}}{L} , \quad \text{large } N , \tag{2.17}
\]

the result obtained earlier for a periodic structure. If instead, we assume that \( 1 \ll N \ll ka^2/L \), the sum over \( \nu \) can be converted to an integral over \( \nu \) from 0 to \( w \) to give

\[
N Z_Y(k) \equiv \frac{(1 + j) \sqrt{\pi ka^2}}{\sqrt{\pi ka^2}} \left[ 1 + \frac{\nu \sqrt{ka}}{\nu \sqrt{ka}} \right] . \tag{2.18}
\]

This limit corresponds to converting the sum over \( \nu \) to an integral in Eq. (2.11), leading to

\[
y_n + \frac{1}{\sqrt{ka}} \sum_{m=1}^{N} \frac{y_m}{\sqrt{\nu_n - \nu_m}} = 1 . \tag{2.19}
\]

For large \( n \), it is easy to show from Eq. (2.19) that the asymptotic form of \( y_n \) is

\[
y_n \to \frac{\sqrt{L}}{\nu \sqrt{ka}} , \tag{2.20}
\]

leading to

\[
N Z_Y(k) \equiv \frac{(1 + j) \sqrt{\pi ka^2}}{\sqrt{\pi ka^2}} \frac{1}{2\sqrt{ka}} \tag{2.21}
\]

This result, which is more accurate than Eq. (2.18) for large \( N \) because it uses \( \sum_{n=1}^{N} y_n \) rather than

\[
\sum_{n=1}^{N} y_n e^{-n/N} ,
\]

suggests that Eq. (2.18) can be made more accurate by replacing the factor \( \nu \sqrt{ka} \) by \( \nu \sqrt{ka}/4 \) to obtain

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This surprising result predicts that the impedance will vary as $N^{1/2}$ once $N > L/g$, and that the transition to the periodic result in Eq. (2.17) takes place when $N > ka/L$.

Finally, we can obtain a result which properly contains both limits by converting the sum over $s$ in Eq. (2.16) to an integral over $j_{g}$ with a lower limit on $j_{g}$ chosen to retain the relation $\sum_{s=1}^{\infty} s^{-2} = 1/4$. In this way we obtain the relation

$$N \bar{Z}_{o} Y(k) = \bar{g}(k) + \alpha \sqrt{N - 1} \tan^{-1} \frac{\alpha}{2\sqrt{N}},$$  (2.23)

with

$$\alpha = \frac{a}{\sqrt{L}},$$  (2.24)

which can easily be seen to give the limit in Eq. (2.17) as $N \gg ka/L$ and the limit in Eq. (2.22) for $1 \ll N \ll ka/L$. The change to $N=1$ in Eq. (2.23) is made to give the correct limit when $N=1$.

We have repeated the analysis for a small obstacle, that is where $kg - 1$ even though $kL > 1$. The entire analysis and final result in Eq. (2.23) are unchanged, except that $F_{o}(k)$ is now the actual single obstacle admittance. In the case $kg \ll 1$, Gluckstern and Neri have shown that

$$F_{o}(k) \equiv \sum_{s=1}^{\infty} \frac{e^{-jb/a}}{b/a + j/\pi},$$  (2.25)

where $\Delta$ is the cross sectional area of the (small) pillbox.

Discussion

Equation (2.23) gives a result for the average impedance (admittance) for $N$ equally spaced identical cavities at high frequency. The transition to the periodic result shows clearly when $NL \gg kg^2$. In addition, Eq. (2.23) predicts that, for $ka' \gg NL$ the impedance will return to a $k^{1/2}$ dependence at high frequency, but with a coefficient which varies as $N$ for large $N$, as given in Eq. (2.22). This has important implications where there are a large number of obstacles, and where conventional wisdom has up to now been to add impedances. We have checked this result by evaluating $y_{n}$ numerically from Eq. (2.19).

In addition, we have allowed $g/L$ and $L$ to be different for each cavity and confirm numerically that the $N^{1/2}$ result does not depend on delicate phase cancellations. Moreover, we expect that the analysis for the transverse coupling impedance will be parallel, and therefore believe that our conclusions are correct at high frequency for multiple obstacles of any shape in a beam pipe of any cross section.

Acknowledgment

I would like to thank Filippo Neri for several helpful conversations, and John Diamond for performing the numerical calculations.

References

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