INVARIANTS OF BETATRON MOTION AND DYNAMICAL APERTURE

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1. Summary and conclusions

In this contribution the approximate limiting stable amplitude for the non-linear betatron motion in circular accelerators containing sextupoles is derived for the purely one dimensional case of the horizontal motion. Using as input the beta functions, the sextupole positions as well as the sextupole strengths and the phase differences between the sextupole positions, the algorithm gives an estimate of the dynamical aperture at a certain observation point in the ring (see Fig. 1). The method used is to derive a generalized invariant from Hamilton's equations which leads to an integral equation for the action \( J(\phi) \). This equation is solved approximately by iteration and the limit of stable motion is determined by calculating the maximum initial displacement \( x_0 \) for which the action \( J \) remains real and positive for all \( \phi \). The agreement between this analysis and tracking experiments in the case of a FODO-lattice with two families of sextupoles and a more general example of a LEP-type structure with insertions is good.

2. Generalized invariants

The differential equation of betatron-motion is taken from Ref. [1] and reads as:

\[
x''(t) + Q^2 x(\phi) \cdot \beta_{5/2}(\phi) \cdot x^2 = 0 \quad (1)
\]

where \( K' \) and \( \beta \) are the sextupole strength and horizontal \( \beta \)-function as functions of the azimuth \( \phi = \phi/\Omega \). This equation can be derived from the Hamiltonian:

\[
H(x,p) = \frac{1}{2} (Q^2 x^2 + p^2) - \frac{Q^2}{3} K'(\phi) \cdot \beta_{5/2}(\phi) \cdot x^3 \quad (2)
\]

with Hamilton's equations:

\[
\frac{dx}{d\phi} = \frac{p}{\Omega} \quad \text{and} \quad \frac{dp}{d\phi} = -\frac{dH}{dx} \quad (3)
\]

Multiplying the second equation (3) by \( p \) and integrating w.r.t. \( \phi \), we obtain the expression:

\[
\frac{1}{2} p^2 + Q^2 \int x \cdot dx + \int x^2 \cdot f(\phi) \cdot d\phi = C \quad (4)
\]

with \( f(\phi) = -Q^2/2 \cdot K'(\phi) \cdot \beta_{5/2}(\phi) \). Now we use the first equation (3) in order to eliminate the explicit \( \phi \)-dependence from Eq. (4) and we find a formal (generalized) invariant, i.e., a constant of the motion depending only on the canonical variables \( x \) and \( p \):

\[
p^2 + Q^2 x^2 + 2 \int x^2 \cdot f(\phi) \cdot dx = C \quad (5)
\]

Finally we perform the usual transformation to action-angle variables [2] as:

\[
x = J^{1/2}(\phi) \cdot \cos \phi \quad (6)
\]

\[
p = Q J^{1/2}(\phi) \cdot \sin \phi \quad (7)
\]

This yields an integral equation for the unknown \( J(\phi) \):

\[
J(\phi) = C - \frac{2}{Q^2} F(\phi, J(\phi)) \quad (8)
\]

\( F \) is the double integral contained in Eq. (5) where \( x \) and \( p \) have been replaced by Eqs. (6) and (7).

3. Iterative solution and stability limit

The method we choose to solve Eq. (8) approximately is to rewrite it as a Picard-type of iteration [3] like:

\[
J_{n+1}(\phi) = C - \frac{2}{Q^2} F(\phi, J_n(\phi)) \quad (9)
\]

As starting solution \( J_0(\phi) \) we choose the \( J \) for the linear equation (1) which is a constant and given by:

\[
J_0 = \frac{2}{Q^2} \quad (10)
\]

Inserting this constant into the iteration (9) we find a first correction due to the non-linear part of Eq. (1) as:

\[
J_1 = J_0 + \frac{2}{Q^2} \int \cos^{3/2} \phi \cdot \sin \phi \cdot f(\phi) \cdot d\phi \quad (11)
\]

We shall denote the integral in Eq. (11) by \( G(\phi) \). From the transformations (6) and (7) and from Eq. (11) we find the retransformation from \( J_0 \) to \( x_0 \) when \( p(\phi) = 0 \):

![Diagram](image-url)
\[-2 \frac{Q}{G(0)} J_0^{3/2} + J_0 \ x_0^2 = 0, \quad (12)\]

where we have chosen the negative sign for \(J_0^{3/2}\).

We are now interested in the maximum stable value \(x_0\) which, in the case of Eq. (1), leads to a bounded motion for all \(\phi\). This limit can be found by investigating the transformation (6) from \(x\) to \(J\). We see that for real \(J\), \(x\) and bounded \(J\) this transformation must describe a bounded motion since the cos-function is bounded. On the other hand \(x\) has to be a real quantity because it represents the real motion in space. Now, if \(J_1^{1/2}(\phi)\) becomes complex for a certain interval of \(\phi\), then \(x\) has to become complex and therefore behaves exponentially. Thus looking for mechanisms which render \(J_1^{1/2}(\phi)\) complex should indicate the limit of bounded motion.

There are two possibilities for \(J_1^{1/2}\) to become complex. The first one comes from the transformation equation (12). This equation depending on the coefficient of \(J_0^{3/2}\) will either have two real solutions, one real double solution, or complex solutions. From Eq. (12) we may derive an exact criterion for the limiting case of one real double solution for \(J_0\):

\[x_0 = \frac{Q^2}{3^{3/2} \ G(0)} . \quad (13)\]

Now suppose we have found a real solution for Eq. (12), then there exists a second possibility for \(J_1^{1/2}\) to become complex (imaginary). This is obvious if we inspect Eq. (11) for \(J_j(\phi)\) which can be written as:

\[J_j(\phi) = J_0 \left[ 1 + \frac{2}{Q^2} \ J_0^{1/2} \ G(\phi) \right] . \quad (14)\]

Whenever for a certain interval of \(\phi\) the expression in the brackets of Eq. (14) becomes negative, then \(J_1^{1/2}\) becomes imaginary and thus \(x\) will be unbounded. The limit is reached when:

\[\min_\phi G(\phi) = -\frac{Q^2}{2 J_0^{3/2}} , \quad (15)\]

and the limiting \(x_0\) follows from Eq. (12).

In Fig. 2 we show \(J_j(\phi)\) for two different \(J_0\) where we used for \(f(\phi) = \cos(\phi/2)\). We see a stable case (3 positive) and the limiting case where the \(J\)-function just touches the \(+\)-axis.

Fig. 2 - Stable and limiting functions \(J_j(\phi)\).
Fig. 3 - Dynamical aperture and tracking results for LEP-type structure at the low-θ insertions function of K21.

References