Abstract

Recent progress in the study of high-current, low-emittance, charged-particle beams may have a significant influence on the design of future linear accelerators and beam-transport systems for higher brightness applications. Three space-charge-induced rms-emittance-growth mechanisms are now well established: (a) charge-density redistribution, (b) kinetic-energy exchange toward equipartitioning, and (c) coherent instabilities driven by periodic focusing systems. We report the results from a numerical simulation study of emittance in a high-current radio-frequency quadrupole (RFQ) linear accelerator, and present a new semiparametric equation for the observed emittance growth, which agrees well with the emittance growth predicted from numerical simulation codes.

Introduction

The problem of obtaining high-current beams with low output-emittance is a challenging one for accelerator design. To solve this problem, we need to understand the limits on maximum beam current and minimum emittance. The current is limited by the focusing available to confine a space-charge defocused beam with finite emittance to within a given radial aperture a. Current-limit formulas have been derived for both continuous beams in periodic focusing systems and for bunched beams in linacs. A deterioration of the beam quality for nonstationary initial beams as a result of rms emittance growth has been observed both in numerical simulation studies and in experimental measurements. This growth can occur even when Liouville's theorem is satisfied and the true phase-space volume of the beam remains invariant. A small emittance is desired not only to avoid a reduction in the beam-current limit, but also for operational reasons because of the desirability of reducing beam halo and particle loss. Furthermore, some applications place severe requirements on focusing the output beam, which can only be obtained by providing a very low emittance beam. Until recently, space-charge-induced emittance growth in linacs could be calculated only by computer simulation, and no analytic predictions were available to serve as guidance for high-current-low-emittance linac design, even for the ideal case of perfectly aligned beams with no nonlinear external fields and no image forces.

Space-Charge-Induced Emittance Growth in RF Linacs and Beam-Transport Systems

A new understanding of the relationship between rms emittance and space-charge field energy has led to some useful approximate equations for emittance growth. The initial suggestion for such a relationship was made to explain numerical simulation results for periodic quadrupole transport. For a round continuous beam in a linear focusing channel, a differential equation relating the time rate of change of the rms emittance and field energy was derived for an arbitrary distribution as

\[ \frac{dE}{dt} = \frac{X^2 K}{2} \frac{dU_{eq}}{dt} \]  

where the 4-rms emittance \( E \) is defined in terms of the second moments of displacement \( x \) and divergence \( x' \) as

\[ E = \left( \frac{1}{2} x^2 x'^2 \right)^{1/2} \]  

The quantity \( K \) is the generalized permeance defined as \( K = e/2 \pi \varepsilon_0 m c^2 \beta^2 e_0^2 \) and is the beam radius of an equivalent uniform beam (a uniform beam, with the same current and same second moments \( r^2, x^2 \), and \( x'^2 \) as the real beam). The dimensionless quantity \( U_{eq} \) is called the nonlinear field energy, proportional to the difference between the space-charge field energies of the real beam and of the equivalent uniform beam. The nonlinear field energy is found to be independent of beam current and rms beam size and depends only on the shape of the charge density in real space. The \( U_{eq} \) minimum is zero (for uniform charge density), and it increases as the charge density becomes more nonuniform. Thus, \( U_{eq} \) is a measure of the nonuniformity of the charge density and furthermore is the field energy that is available for emittance growth.

Space-Charge-Induced Emittance Growth in RF Linacs

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A generalized form of Eq. (1) for a bunched beam was derived by Hofmann and can be written for three degrees of freedom \( x, y, \) and \( z \) (with linear focusing in each plane) as

\[ \frac{1}{2} \frac{dE^2}{dt} + \frac{1}{2} \frac{dE^2}{dt} = -2 \frac{d(W - W_{eq})}{dt} \]  

where \( N \) is the number of particles in the bunch, and \( W \) and \( W_{eq} \) are the space-charge field energies of the real beam and of the equivalent uniform beam. Equation (2) [and Eq. (1)] can be integrated for the case of an rms-matched space-charge-dominated beam with linear continuous focusing (smooth approximation for periodic focusing) because the rms beam sizes are approximately constant, independent of emittance. Equations for 4-rms emittance can be derived for both bunched and continuous nonstationary beams. The result for an axially symmetric bunched beam in the transverse plane can be written

\[ E_{x}^{2} = E_{y}^{2} = \frac{9 + P_{x} + P_{y}}{2 + P_{x} - P_{y}} \frac{16 G_{x} C_{x}}{P_{y} - P_{x}} \left( \frac{K_{x}^{2} f_{x}^{2}}{a_{x}^{2}} \right)^{23} \left( U_{nf} - U_{ns} \right) \]  

and for the longitudinal plane the result is

\[ E_{z}^{2} = E_{z}^{2} = \frac{2 + P_{z} - P_{y}}{a_{z}^{2} (2 + P_{z} - P_{y})} \frac{16 G_{z} C_{z}}{P_{y} - P_{x}} \left( \frac{K_{z}^{2} f_{z}^{2}}{a_{z}^{2}} \right)^{23} \left( U_{nf} - U_{ns} \right) \]  

where the subscripts \( x, y, \) and \( z \) refer to the initial and final states of the beam, \( a \) and \( b \) are the rms beam sizes in \( x \) and \( z \), and \( G_{x}(b/a) \) and \( G_{z}(b/a) \) are bunch geometry factors equal to unity for a spherical bunch. These factors are given approximately as \( G_{x}(r) = (91 - 1/r^2)^{1/2} \) for \( 13r < 2 \) and \( 2(r - 1/2)^{1/2} \) for \( 2 < r < 13 \). The quantities \( P_{x} \) and \( P_{y} \) are zero-current phase advances of the transverse to longitudinal oscillations per focusing period \( L \). The quantity \( P \), called the partition parameter, is defined as \( P = 2 \pi^{2} x^2 \) and is a nonrelativistic measure of the kinetic-energy asymmetry in the rest frame of the bunch. By analogy with the continuous beam, we have defined a bunched-beam particle current in terms of the number of particles \( N \) per bunch as \( K_{y} = e^{2} N / 2 \pi \varepsilon_0 m c^2 \beta^2 e_0^2 \). For a bunched beam with current \( I \) (average over one rf cycle during the beam pulse) in a linac with rf wavelength \( \lambda \), \( N \) is given by \( N = 1/\ell \). Two mechanisms contribute to the emittance growth in Eqs. (3a) and (3b): (1) kinetic-energy exchange between the longitudinal and transverse planes, which is zero only
when $P$ does not change, and (2) charge redistribution, which is zero only when $U_m$ does not change and corresponds to an exchange between field energy and particle kinetic energy as the charge density in real space evolves. It is interesting that Eqs. (3a) and (3b) can also be derived from energy conservation in the rest frame for a space-charge-dominated beam.

Although the initial values of the parameters $P$ and $U_m$ are known in principle for a given initial state of the beam, we have no theory available to allow a determination of the time dependence for $P$ and $U_m$. However, numerical simulation studies have shown that for highly space-charge-dominated beams in linear focusing channels, the charge density approaches a nearly uniform distribution ($U_m \approx 0$) and the beam tends to an equipartition state ($P = 1$). The time scales for these mechanisms are different; charge redistribution is very fast and can occur within about a plasma period, whereas the slower kinetic-energy exchange process can take from a few to tens of plasma periods. The final uniform charge density may be explained as a tendency for charge redistribution to shield the interior of the beam from the linear external force, in analogy with a cold plasma. This generally results in a matched charge-density profile consisting of a uniform central core with a Debye sheath at the beam edge, whose thickness is given by the Debye length and which becomes zero in the extreme space-charge (EC) limit, resulting in a uniform density. Why a beam, whose interactions predominately occur through collective fields rather than collisions, should equipartition is not yet clear. Nevertheless, these assumptions about the final state of the beam can provide us with a model for predicting final emittance growth. Numerical simulation results in continuous linear focusing channels for bunched beams are in good agreement with Eqs. (3a) and (3b) and for continuous beams with corresponding emittance growth equations.

The emittance growth formulas presented above have been derived for continuous linear focusing. It is of great interest to determine whether the equations do represent a good smooth approximation to emittance growth for periodic focusing, such as is used in real linacs. In addition, an initially uniform charge density in real space ($U_m = 0$) eliminates emittance growth from charge redistribution in continuous focusing systems, and it is important to determine whether this conclusion is also valid for periodic systems.

In fact, the published numerical studies do appear to support the conclusion that initially uniform beams in quadrupole periodic channels give approximately no emittance growth, at least for $\alpha_{ox} \leq 60^\circ$ to $60^\circ$. In addition, for $\alpha_{ox} < 60^\circ$, the emittance growth formulas for charge redistribution also seem to represent a very good approximation. For $\alpha_{ox} > 90^\circ$, significant deviations are observed from the formulas, and additional emittance growth is observed for both initially uniform and nonuniform beams. These results appear consistent with the interpretation that for certain cases such as $\alpha_{ox} < 90^\circ$ in periodic channels, another mechanism becomes important namely coherent instabilities driven by the periodic structure, which have been studied in detail for the Kapchinskii-Vladimirskii (K-V) distribution.

Experimental support for the validity of the charge-redistribution equation for emittance growth in periodic quadrupole beam-storage systems can be seen in the published results of the Gesellschaft f"ur Schwerionenforschung (GSI) experiment with an initial quasi-Gaussian beam, and in the Lawrence Berkeley Laboratory (LBL) experiment with an initial quasi-uniform charge density. The experimentally measured emittance growth data from the two experiments at low phase-advance (high space-charge) values are significantly different from each other, and both are in agreement with the prediction of the emittance-growth equation.

### Study of Emittance Growth in a High-Current RFQ Linac

We have conducted a study of emittance growth in the RFQ for a 353-MHz RFQ linac, which bunches and accelerates an initial 100-keV $H^+$ beam to 3 MeV in 262 cells of length $\beta \Delta /2$. The synchronizing phase is ramped from $-90^\circ$ to $35^\circ$ and the longitudinal and transverse current limits, calculated at the end of the gentle (adiabatic) buncher section are 175 mA. The zero-current phase advance $\alpha_{ox}$ ranges from an initial value of $34^\circ$ to a minimum value of $30^\circ$ at the end of the gentle buncher, which implies that coherent instabilities are not expected to play a significant role. Numerical simulation studies were conducted with the code PARMTEQ using 3600 particles per run. The design results in high transmission; the transmission value exceeds 90% at 100 mA and exceeds 82% for currents as large as 165 mA. Figure 1 shows the transmission and transverse 4-rms normalized final emittance versus input current for two different input emittances from numerical simulation studies of an $H^+$ RFQ described in the text. Smooth curves are drawn through the points shown.

In Fig. 2, we show the transverse 4-rms normalized emittance (defined as $\epsilon = E_{xy}$) for the transmitted beam (averaged over $x$ and $y$) versus initial beam current for two different input emittances. We used an initial 4-D Waterbag distribution in transverse space (uniform filling of a 4-D hyperellipsoid volume) and a uniform filling in longitudinal position with zero energy spread. At zero beam current, there is almost no growth of emittance, consistent with the hypothesis that the observed growth at nonzero beam currents is caused by space-charge forces. For both input emittances, the final emittance rises with current to a peak near 30 to 50 mA. We see that the emittance growth is not a simple monotonic function of beam current, and is rather insensitive to beam current between about 20% and 90% of the calculated-current limit. Most of the particle losses appear to result from longitudinal effects; thus, the behavior of the transmission is nearly the same for both input emittances, and we believe that particle-loss effects are inadequate to explain the emittance curves of Fig. 1.

In Fig. 2, we show the transverse 4-rms normalized emittance versus cell number for input current 55 mA, input emittance 0.020 n$^2$cm$^2$mm$^2$, and an initial Waterbag transverse distribution. Figure 2 shows that the emittance growth occurs predominately between cells 20 and 120, while the beam is being bunched. Also in Fig. 2, we show the longitudinal rms half-length $z_{rms} = \sqrt{z^2}$.
distributions. Further examination shows that charge emittance is nearly the same for the three different initial distributions. The charge redistribution occurs very rapidly after the dc beams enter the RFQ, resulting in a nonuniform transverse density distribution (hollow at some locations) that is nearly the same for the three different initial distributions. This charge redistribution results in a small, rapid emittance change that is largest for the initial Gaussian beam. However, after the emittance growth during the bunching, these small differences in charge distribution and transverse emittance appear to nearly vanish. In the absence of emittance growth, the results would lie along the 45° line shown in Fig. 4. Instead, the results show a lower limit on final emittance as the initial emittance decreases to zero, a phenomenon that was first reported in numerical studies of space-charge effects in drift-tube linacs. The fitted curve shown in Fig. 4 is from the semi-empirical equation that will be discussed in the next section.

We now summarize the main features observed in our numerical study of RFQ transverse emittance for a typical Los Alamos design: (1) the emittance growth observed in the PARMTEQ numerical simulations is predominantly caused by space-charge forces, (2) most of the growth occurs in the initial bunching section and is a strong function of the longitudinal beam size, (3) the growth is insensitive to beam current for currents in excess of about 15-20% of the current limit (an effect for which bunching may be responsible), (4) the growth is insensitive to bunching from continuous to bunched, and where particle losses can affect the results. Nevertheless, we are encouraged to begin our search for a better quantitative description of RFQ transverse-emittance growth by attempting to develop a model and a semi-empirical equation based on Eq. 3(b).

We postulate that the geometry factor G(b/a) (which increases with increased bunching, or smaller b/a) is a parameter that is not stationary in the RFQ because of the longitudinal fields, which are present even at the input. Figure 4 shows that the final transverse emittance at zero, a phenomenon that was first reported in numerical studies of space-charge effects in drift-tube linacs.
function of the ratio $I/I_1$, where $I$ is the current and $I_1$ is the current limit. To account for the insensitivity of emittance growth to beam current for a given design, we postulate that $G_i(b/a)_{1/3}$ is independent of $I/I_1$. We propose this as an approximation, valid for $0.2 \leq I/I_1 \leq 0.9$. $U_{nf}$, and the magnitude of $G_i(b/a)$ are functions only of the detailed design procedure. Using these assumptions, Eq. 3a can be written for the 4-rms normalized emittance as

$$\epsilon_r = a_1 \epsilon_r^2 + a_2 \frac{1}{(A/a)^{1/3}},$$

(4)

where $I$ is the (electrical) current limit in amperes (for Los Alamos RFQ designs, $I$ represents equal limits for both the longitudinal and transverse planes), $\sigma_{max}$ in radians is the zero-current phase advance per $b$ at injection (after radial matching), $\lambda$ is the rf wavelength in centimeters, $q$ is the charge state, and $A$ is the mass number in atomic mass units.

The smooth approximation formula gives $\sigma_{max} = (q/A)(eVmc^2)/(\lambda/\sigma_{max})$, which corresponds to cm-mrad units for the 4-rms normalized emittance. These results and also closely represent the results for a variety of other recent designs using the latest Los Alamos design procedures.

The second term in Eq. (4) depends on $\lambda^2$, and this strong dependence is modified only slightly in the Los Alamos design procedure when $\sigma_{max}$ (which also depends on $b$) is limited by the Kilpatrick electric-breakdown criterion.$^{22}$ As part of an overall test of the scaling with respect to these parameters, we have compared the predictions of Eq. (4) with the results of PARMTEQ numerical studies using 360 particles per run of RFQ designs done several years ago for heavy-ion fusion applications.$^{23}$

In Fig. 5, we show the final 4-rms, normalized transverse emittance versus frequency for $^{20}$Ne$^{+1}$ and $^{238}$U$^{+1}$ designs, all with injection energy 0.4 MeV and final energy 4.0 MeV. The 4-rms, normalized input emittance was 0.0132 n-cm-mrad for all designs and the input current was set equal to one-half the calculated-current limit. The current limits varied from 20 mA to 2 A, with the highest current limits at the lowest frequencies. The PARMTEQ results for $^{20}$Ne are shown in Fig. 4 and compared with the results of Eq. (4) using numerical values $a_1 = 0.75$ and $a_2 = 3.2 \times 10^8$ mrad$^2/(A/amu)^{1/3}$. These values of $a_1$ and $a_2$ result in a good agreement with a large variety of Los Alamos designs for the procedures used several years ago. For the $^{20}$Ne results in Fig. 5, Eq. (4) compares closely with numerical calculations of the emittance growth over a range of two orders of magnitude in emittance, as the frequency varies by one order of magnitude, and the current-limit values (not shown) vary by nearly two orders of magnitude. Then the same equation gives a close prediction for the $^{238}$U results, also shown in Fig. 5. Other simulation studies have been made to test the current-limit dependence of Eq. (4) at fixed values of $\lambda$ and $q/A$ and to confirm the prediction of Eq. (4) that the final emittance has no explicit dependence on $\beta$ (but does depend on $\beta$ through the current limit).
The rf wavelength dependence results from the fact that at fixed current, the number of particles per bunch increases in proportion to $A$. The lack of explicit dependence on $B$ implies that at fixed-current limit, the emittance growth is independent of RFQ injection energy. The existence of a minimum final emittance at small input-emittance values is a consequence of a transfer of field energy to kinetic energy as the beam bunching increases the space-charge force. The coefficient $a_1$ in Eq. (4) is expected to be a strong function of the bunching. Consequently, one expects that when the first term of Eq. (4) is small (small input emittance) compared to the second, weaker bunching forces may be necessary to avoid the growth of emittance. Likewise, when the first term of Eq. (4) is large (large input emittance), stronger bunching forces may be tolerable. The RFQ design improvements based on these ideas are currently being studied. We plan to conduct further studies to follow the evolution of the $U_n$ and $P_n$ parameters of the beam to obtain more information about the details of the emittance growth process.

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References