certain coherent displacements did indeed linearly perturb the dispersion. This observation led to several studies of systematic and random errors, with the conclusion that this effect stems from the fact that the dispersion in the SLC is not always matched (especially in the vertical plane) due to the necessity of rolling the achromats about the beam axis to follow the site terrain. We will now review these findings in simple form.

Non-Dispersive Condition

Consider a differential slice of an Arc magnet with length \( ds \). The transverse field components can be expressed as follows,

\[
B_y = B_{yo} + B_{yo}' x + \frac{1}{2} B_{yo}'' (x^2 - y^2),
\]

\[
B_x = B_{xo}' y + B_{xo}'' xy,
\]

where the coefficients \( B_{yo}, B_{yo}', B_{xo}' \) and \( B_{xo}'' \) represent the dipole, quadrupole and sextupole strengths of the combined function magnet, respectively. Here the prime indicates differentiation with respect to transverse displacement.

Suppose at this location that there is no dispersion in the \( y \) plane and that the horizontal dispersion is equal to \( \eta \). Suppose further that the element magnet is displaced from the reference axis by \( \Delta x \) and \( \Delta y \). Then, for a momentum \( p = p_0 (1 + \delta) \),

\[
x = \eta \delta + \Delta x,
\]

\[
y = \Delta y.
\]

With these equations inserted in Eqs. (1) and (2), the angular "kicks" \( dB \) in horizontal and \( \Delta y \) in vertical planes of a ray passing through the elemental magnet may be written as follows,

\[
d\theta = -\frac{ds}{p_0 (1 + \delta)} \left\{ B_y (1 + B_{yo}' \delta) + \frac{1}{2} B_{yo}'' \delta^2 \right\} + \left\{ B_x + B_{xo}' \delta \right\} \Delta x + \frac{1}{2} B_{xo}'' \Delta x \Delta y^2,
\]

\[
d\phi = -\frac{ds}{p_0 (1 + \delta)} \left\{ (B_y + B_{yo}' \delta) \Delta y + B_{xo}' \Delta x \Delta y \right\}.
\]

Consider only the terms up to the linear approximation in \( \Delta x \) and \( \Delta y \) and neglect all others, then

\[
d\theta \approx -\frac{ds}{p_0 (1 + \delta)} \left\{ B_y (1 + B_{yo}' \delta) + B_x \right\} \Delta x,
\]

\[
d\phi \approx -\frac{ds}{p_0 (1 + \delta)} \left\{ B_x \Delta y \right\}.
\]

Clearly, the momentum dependence in both equations will factor out if the following conditions are satisfied everywhere,

\[
\frac{B_{yo}'}{B_y} = 1 \quad \text{and} \quad \frac{B_{xo}'}{B_x} = 1.
\]
Eqs. (9) cannot be satisfied at all points since within a magnet \( n \) varies while \( B, L^2 \), and \( B_r \) do not. However, if \( n \) is averaged over a single Arc magnet the second equation in Eqs. (9) is satisfied, i.e.,

\[
\frac{B'_o \bar{n}}{B_o} \bigg|_{\text{single magnet}} = 1 .
\]

Eq. (10) is the non-dispersive condition given in Ref. 2. This equation is satisfied by the fields of combined function magnets in second order achromats.

Eqs. (7), (8), and (10) can be used to conclude that in the linear approximation the matched dispersion function \( \eta \) is immune to effects due to random or purposely induced transverse displacements of single magnets in an achromat composed of combined function magnets. For the SLC Arcs a transverse displacement of 100 microns for \( \Delta x (\Delta y) \) steers a beam horizontally (vertically) with a strength equivalent to 1.2% of the dipole bending field \( B_o \) making such displacements an effective method of steering. Furthermore, since this non-dispersive property applies to a single magnet such a magnet can be used to provide a non-dispersive "kick" anywhere in any lattice provided that the average value of the dispersion is not zero.

Relationship of Field Components

The first equation in Eqs. (9) is also satisfied when \( n \) is averaged over each magnet of an entire cell, i.e.,

\[
\frac{B'_o \bar{n}}{B_o} \bigg|_{\text{entire cell}} = 1 .
\]

This is shown to follow from the definition of the \( \eta \) function and a general integral equation derived in Ref. 9 and which apply to a cell of an arbitrary lattice. Eqs. (10) and (11) can be used to derive relationships for the field components correct to a few percent for achromats composed of combined function magnets. These relationships are useful because for such achromats \( \eta \) can be integrated analytically as in Ref. 9. Using the SLC Arc cell for an example, Eqs. (10) and (11) become

\[
\frac{B'_o \bar{n}_F}{B_o} = 1 , \quad \frac{B''_o \bar{n}_D}{B_o} = 1 ,
\]

and

\[
\frac{|B'_o|}{2B_o(\bar{n}_F - \bar{n}_D)} - 1 ,
\]

where the subscripts \( F \) and \( D \) refer to the focusing and defocusing magnets, respectively. Similar relationships can be written for combined function lattices of second order achromats composed of magnets with differing lengths and field components.

Effect of Mismatched Dispersion

Equations (3) and (4) are now rewritten to allow the position in the differential slice to include deviations from the matched dispersion. Let

\[
x = \Delta x + \eta_x \delta \equiv \Delta x + (\eta_x + \Delta \eta_x) \delta ,
\]

and

\[
y = \Delta y + \eta_y \delta \equiv \Delta y + \Delta \eta_y \delta ,
\]

where \( \Delta x, \Delta y \) are transverse displacements, \( \eta_x, \eta_y \) are the actual dispersion functions, \( \eta_x, \eta_y \) are the matched \( \eta \)-function and \( \Delta \eta_x \) and \( \Delta \eta_y \) are the differences between the actual and matched functions.

Again, Eqs. (14) and (15) are inserted in Eqs. (1) and (2). Then retaining only terms linear in \( \Delta x \) and \( \Delta y \), averaging the dispersion over the length of the moved magnet, \( l \), and separating the remaining terms in dispersive and non-dispersive groups, one finds

\[
\Delta \phi \approx -\frac{l}{p_o(1 + \delta)} \left\{ \left( B'_o + B''_o \bar{n}_o \delta \right) \Delta x + B''_o \bar{n}_o \delta \left( \Delta \eta_x \Delta x - \Delta \eta_x \Delta y \right) \right\}
\]

\[
\Delta \phi \approx -\frac{l}{p_o(1 + \delta)} \left\{ \left( B'_o + B''_o \bar{n}_o \delta \right) \Delta y + B''_o \bar{n}_o \delta \left( \Delta \eta_y \Delta y + \Delta \eta_y \Delta x \right) \right\}
\]

Inserting Eq. (10) and taking the derivative with respect to \( \delta \) at the point \( \delta = 0 \), the anomalous \( \eta'_x, \eta'_y \), for \( x \) and \( y \) planes, respectively, become

\[
\Delta \eta'_x = -\frac{B''_o}{p_o} \left( \Delta \eta_x \Delta x - \Delta \eta_x \Delta y \right)
\]

\[
\Delta \eta'_y = \frac{B''_o}{p_o} \left( \Delta \eta_y \Delta y + \Delta \eta_y \Delta x \right)
\]

These equations show that in the presence of a magnet misalignment a deviation of the dispersion from that of the matched dispersion will generate an anomalous \( \eta'_x, \eta'_y \). This is important to the SLC Arcs in that the dispersion is not always matched due to the required rolls. The consequence is that the Arcs are more sensitive to misalignment than if they had been constructed flat. For a typical 10° roll as in the Arcs, \( \Delta \eta_x \) is small compared to \( \Delta \eta_y \) so the main contribution to both \( \Delta \eta'_x, \eta'_y \) comes from the terms proportional to \( \Delta \eta_y \). That means that \( \Delta \eta'_x \) is produced by a vertical displacement \( \Delta y \) and vice versa. On the other hand, for the whole Arc the accumulated error in \( \Delta \eta_x \) becomes large and both terms in Eqs. (18) and (19) are significant.

Computer Simulations

The predictions for the behavior of anomalous dispersion given in Eqs. (18) and (19) have been compared with computer simulations of both rolled and unrolled achromats. For the displacement of a single magnet, the results are in good agreement. That is, for an unrolled achromat (matched dispersion) there is no effect, whereas for a rolled achromat, the induced anomalous dispersion, as calculated using Eqs. (18) and (19), agree quantitatively with the simulation results. This agreement was confirmed for several magnets located at arbitrary points where the mismatched dispersion was either large or small in magnitude.

Systematic Translations of Magnets

With agreement established for simple cases, computer simulations were now used to investigate the effects of systematic errors in whole achromats and finally for the entire north arc. Systematic translations of magnets in the arcs can have many causes; here we will examine two particular ones. In the arcs there is a one-to-one correspondence between magnet movers (steering correctors) and beam position monitors (BPM's). In this scheme each focusing magnet is moved horizontally (vertically) to steer the beam through the BPM's attached to the next focusing (defocusing) magnet. This scheme
utilizes 10 correctors and 10 monitors in the horizontal plane and 10 correctors and 10 monitors in the vertical plane per achromatic. The position monitors are placed in the drifts between magnets. Thus the matched dispersion has the same magnitude ($\eta_{BPM} \sim 35$mm) at all BPM's. Therefore, either a) a systematic error in BPM alignment or signal processing or b) steering a beam which has the wrong energy with respect to the arc excitation will cause a systematic offset of all horizontally focusing magnets. A systematic BPM error will do the same in the vertical plane. Fortunately, these systematics will show up in the harmonic analysis of the magnet mover positions and, in principle, is correctable for magnitudes greater than $\sim 30$ microns. Steering an off-momentum beam with $\Delta P = 10^{-3}$ will induce a systematic error in the magnet positions of 28 microns which is just detectable.

In Figures 1 and 2 $\Delta \eta_2$ is shown versus a systematic misalignment of BPM's or relative momentum error. Here, $\Delta \eta_2 = \eta_2 - \eta_0$ where $\eta_0 = 47$mm at the end of the north arc.

In Figure 1, each BPM is offset by the same amount and the arc steering algorithm applied. Because of the one-to-one correspondence between each BPM and the upstream magnet mover a systematic translation is introduced at every horizontally focusing magnet. A systematic offset of 100 microns at the BPM's will cause an equivalent offset of $\sim 50$ microns at each horizontally focusing magnet. This same relationship holds for the vertical plane. It can be seen in Figure 1 that a systematic displacement of this magnitude will cause a $50 - 80\%$ change in the horizontal dispersion function $\eta_2$ at the end of the north arc.

Next, it can be seen that the results shown in Figure 2 are similar to those shown in Figure 1. Here a beam of the wrong momentum with respect to the arc excitation was steered through the system. In this case, prior to the application of the steering algorithm, the centroid of the beam would have the same offset at each BPM. The action of forcing this centroid offset to become zero at each BPM again introduces a systematic offset of the horizontally focusing magnets. Thus steering a beam with a relative momentum error of $3 \times 10^{-3}$ is equivalent to a systematic error of $\sim 100$ microns at the BPM's. The vertical dispersion $\eta_3$ is affected nonlinearly in both of the above cases, changing by $\sim 12$mm (compared to its nominal value of zero) for a BPM offset of 100 microns.

A systematic error in the position of the vertical BPM's will also cause dispersion changes but at a level reduced by an order of magnitude.

**Conclusion**

In the linear approximation the matched dispersion function is not affected by random or coherent displacements of magnets in a second order achromat made up of combined function magnets. For the SLC Arcs the necessity of rolling the achromats has caused a dispersion mismatch which has increased the sensitivity to transverse errors. In particular, the increased sensitivity to systematic errors will require close attention to these errors in component alignment and signal processing electronics for both the beam position monitors and the orbit correcting magnet movers.

**References**

5. G. Fischer et al., paper submitted to this Conference.

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