NON-LINEAR BEAM-BEAM FORCES IN STORAGE RINGS.
PARTICULAR ANALYSIS FOR S.S.C AND L.H.C.
PARAMETERS.

M.Pusterla - Phys. Dept. Università di Padova
and
INFN - Sezione di Padova -Italy
G. Servizi and G. Turchetti - Phys. Dept.
Università di Bologna
and
INFN - Sezione di Padova -Italy

Theoretical models, suitable for description of the long behaviour of bunched and unbunched beams of particles in accelerators and storage rings, are becoming more and more appreciated by physicists that want a high luminosity joined with the stability of the beams. Such a point is going to be particularly important for the next generation machines as L.E.P., S.S.C. and L.H.C.

In this note we are giving a simplified analysis of the beam-beam non-linear effects for proton colliders on the basis of the latest designs (we think of S.S.C. and L.H.C.). Before doing that, however, we like to consider the general features of the dynamical approaches that describe the beam-beam forces both for the proton-proton rings (fixed angle collision) and for proton-antiproton or electron-positron rings (head-on collisions): they follow directly from the recent developments of non-linear classical mechanics, namely the K.A.M. theorem and the transition to a chaotic motion in deterministic mechanical systems.

We here consider the weak-strong description of the beam-beam forces where a test particle of a certain beam gets periodically perturbed by (non-linear) "kicks" due to the beam-beam collision in the intersection region; the periodic instantaneous kicks have an intensity strictly connected with the charge distribution in the strong beam and modify the momentum of the test particle.

In a proton-proton collider (fixed angle), under suitable geometrical conditions, the betatron "kicks" influences only the vertical (y-axis) betatron motion and one easily obtains the following map:

\[
\frac{Y'_{n+1}}{Y_{n+1}} = \frac{1}{\gamma} \left[ Y_{n+1} - \Phi(Y_{n+1}) \right] + S \quad (1)
\]

where \( \gamma \) is the vertical tune, \( C = \cos(\gamma \Theta) \),

\[ S = \sin(\gamma \Theta) \Theta - \cdots, \quad m = \text{being the number of interactions of the test particle per revolution} \]

and \( Y' \) is the left-derivative versus the revolution angle \( \Theta = \omega_n t \), \( \omega_n = 2\pi / T_n = \text{cyclic revolution frequency} \); finally \( \Phi(y) \) is the function \( \int_0^y \frac{dt}{\gamma y/4} \cdot \exp(-t^2) \) that comes into the map because of gaussian charge distribution in the strong-beam.

Map general properties and Birkhoff's analysis. Eqs. (1), (1') give a simplicistic, area preserving, map which can be rewritten, after rescaling the variables \( y = Y_n/\alpha y \), \( y' = Y'_n/\alpha y' \):

\[
\frac{Y_{n+1}}{Y'_n} = A \frac{Y_{n+1}}{Y'_n} - \frac{\gamma}{\gamma'} F(Y_n) \frac{\sin(\gamma \Theta) v}{\cos(\gamma \Theta) v'} \quad (2)
\]

with \( F(Y) \) defined as follows:

\[
\gamma [F(Y) + \gamma'] (-\gamma 2 y') = -\gamma Y \Phi(Y) \quad (3)
\]

The eigenvalues \( \lambda_s \) of \( A \) are either real, thus making the origin an unstable hyperbolic fixed point, or imaginary with modulus 1 providing an elliptic (nearly stable) point for the origin: \( \lambda_1 = e^{i\beta}, \cos \alpha = \cos \beta \gamma \gamma' \sin (2\lambda 2 y') \sin \gamma \Theta. \)

Eqs. (1), (1') can be written in complex form:

\[
Z_{n+1} = e^{i\alpha} Z_n + i \frac{\gamma}{\gamma'} \sin \beta \Theta \gamma' \quad (4)
\]

where

\[
Z = \frac{1}{\sqrt{2} \sigma} \left[ Y + i \frac{\gamma}{\gamma'} \sin \beta \Theta \gamma' \left( \frac{1}{2} Y' \right) \right] \quad (5)
\]

We introduce the Birkhoff transformation to the normal form:

\[
Z - \phi(\delta, \phi') = Z - \phi(\delta, \phi') \psi(\delta, \phi') \psi(\delta, \phi') = 1 \quad (6)
\]

Such that in the new variables, eq.(4) has the form

\[
\xi_{n+1} = e^{i\xi \phi(\xi, \phi)} \frac{\xi}{\xi} \quad (7)
\]

If this is possible all points of the map lie on the circles \( |\xi| = 0 \) and in polar coordinates \( \xi = \rho e^{i\phi} \) eq.(7) means:

\[
\xi_{n+1} = \xi_n \quad (8)
\]

\[
\phi_{n+1} = \phi_n + \Omega(\rho) \quad (8')
\]

For the integrable map (8), (8') the transformation eqs.(6) are convergent series whereas for non-integrable maps they are asymptotic series. The singularities of these series are associated with resonances and although not converging any where...
(since it was proved that the functions are analytic nowhere in this case) their asymptotic character can be exploited. The computational procedure is a recurrence for the \( \Phi_k (\xi, \eta) \) polynomials of degree \( K \) and \( 2K \) for \( \Omega \) where

\[
\begin{align*}
\Phi_0 &= \xi, \\
\Phi_n &= \sum_{k=0}^{n-1} \Phi_k \Phi_{n-k-1}, \\
\Omega &= \alpha + \sum_k \xi^k \overline{\eta}^k 
\end{align*}
\]

and a code can be worked out to calculate \( \Omega \) explicitly. The criterion of the ratio applied to the series \((9)\), \((9')\), \((9'')\) shows that they get very close to a piecewise geometric series. One then determines the border of the region beyond which strong instabilities appear. Within this region, by evaluating a certain, sufficiently low, number of terms, one can reproduce the direct map:

\[
\tilde{Z}_n = \tilde{\Psi}(e^{\alpha \xi} \tilde{Z}_0, e^{-\alpha \eta} \tilde{Z}_0^*) \tag{10}
\]

where \( \tilde{\Psi}, \tilde{\Phi} \) represent polynomials from the series truncated at the mentioned order and reproduce quite closely the initial map iterated \( n \) times with the initial condition \( \tilde{Z}_0 = \tilde{\Psi}(\tilde{Z}_0^*, \tilde{Z}_0^*) \).

The asymptotic character of the approximation is seen immediately from the discrepancy \( \delta_n = |\tilde{Z}_n - Z_n|, N \) being the truncation order, \( n \) the iteration order. The result is \( \delta_n \sim n(N/\Pi)^{n^2} \) \( H \) being a constant.

Dissipative beam-beam map (electron-positron rings): The betatron and synchrotron oscillation equations, since we now have beam-beam effects coupled along the three axes of motion \( x, y, z \), are of the type:

\[
\ddot{Z} + 2\omega Z + \omega^2 Z = f(t, x, y, z, s) \tag{11}
\]

where \( Z \) is one of the \( x, y, z \) variables. We simplify the problem by assuming \( f(t, x, y, z, s) = f(t, z) \) and the transformation \( Z = e^{-\beta t}, Y = \frac{\dot{Z} - \beta Z}{\omega} \).

This equation can be derived from a time dependent hamiltonian: since \( f(t, z) = -\frac{\partial}{\partial z} H(z, t) \) \( (13) \)

\[
f(e^{-\beta t} x, t) = -\frac{\partial}{\partial z} H(e^{-\beta t} x, y(t)) \times e^{\frac{\beta}{\omega} t} \tag{13'}
\]

we have the hamiltonian \( H(p, y, t) = + - p^2 + \frac{y'}{2} + \frac{y''}{2} + e^{\frac{\beta}{\omega} t} V(e^{-\beta t} y(t)). \)

In accelerators: \( f(x, t) = f(x) \times S_0(t-\tau) \) and the simplectic (non-autonomous) map is

\[
\begin{align*}
\begin{bmatrix} y_{n+1} \\ y_n' \end{bmatrix} &= \begin{bmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{bmatrix} \begin{bmatrix} y_n \\ e^{\alpha \eta} \end{bmatrix} - \begin{bmatrix} 0 \\ f(e^{-\alpha n} y_n) \end{bmatrix} \\

\end{align*}
\]

We therefore can enlarge the degrees of freedom and make eq.\((14)\) autonomous:

\[
\begin{align*}
\eta_{n+1} &= e^{\alpha n} \eta_n \\

\chi_{n+1} &= e^{\alpha \xi} \left[ \eta_n + \frac{g(\eta_n, \xi)}{\Omega_{n+1}} \right] \\
\end{align*}
\]

where \( g(x) = 2V(x) - V'(x) = xf(x) - 2 \int_0^1 f(r) dr. \)

This approach allows the application of the formalism of simplectic maps to dissipative systems. In particular one can use the Lie-algebra approach and Birkhoff's.

The direct dissipative map is obviously autonomous:

\[
\begin{align*}
\begin{bmatrix} \chi_{n+1} \\ \eta_{n+1} \end{bmatrix} &= \begin{bmatrix} \cos \beta T & \sin \beta T \\\n-\sin \beta T & \cos \beta T \end{bmatrix} \begin{bmatrix} \chi_n \\ \eta_n + 1 \end{bmatrix} \begin{bmatrix} f(\eta_n) \\ -\frac{\beta}{\gamma} \sin T \end{bmatrix} \tag{15}
\end{align*}
\]

Numerical results for the L.H.C. and S.S.C. designs and explanation of tables and figures. We here consider one of the various options recently presented for the proton-proton future colliders L.H.C. and S.S.C. More precisely we choose the following sets of beam-beam parameters:

S.S.C.- vertical tune \( y = 130.2914 \) 6 interaction per revolution

L.H.C.-vertical tune \( y = 78.19 \) 6 interaction per revolution

Figures and tables are given assuming as an independent variable \( X = Y/42 \). Y being the vertical position of the test particle; \( X \) is chosen within the interval (0.1); Birkhoff's series are evaluated in the complex variable \( Z \) defined by eq.(5) and are written as \( \Phi(z), \Psi(z) \) and \( \Omega(z) \), maximum truncation order is \( K_{\text{max}} = 15 \).
The direct map has a polynomial term of degree 15, as a non-linear part, which coincides with the series expansion of the function $2/\sqrt{x} \text{erf}(X)-X$ and is indicated in the figures as $F(X)$. The iterates are named $F_n(Z)$. Table 1 refers to S.S.C., Table 2 to CERN L.H.C. design values obtained from the initial condition $X$ of the first column. Second column named "Rm_{in}(B)" is the average between $R_{min}(B)$ and $R_{max}(B)$. The last columns "R_{max}-R_{min}/R_{med}" gives the percentage difference between $R_{min}$ and $R_{max}$, $R_{min}$ and $R_{max}$ being the minimum and maximum vertical distance of the test particle from the origin respectively. Columns labelled by D give the minimum and maximum distance, from the origin, of the points iterated from the direct map 104 times; those labelled by B are the corresponding values from the Birkhoff series $\Phi(z)$ applied on 1000 points equally spaced on the circle $|\Psi(Z(X))|$. One may notice that the agreement is spectacular.

Fig. 1 and 1' show in logarithmic scales on both axis the percentage difference on 590 iterations between the exact direct map and the Birkhoff $\Phi(z)$ with truncation at $K=3,5,9,11,15$ of the serie $\Phi(z)$ versus the variable $x/2\sqrt{z}$.

Fig. 2 and 2' show in semi-logarithmic scale, the function $|\Psi(Z)|/(|R_{max}|/|R_{min}|)$. We continue the average over 100 iterations; the dotted line gives the same quantity calculated from the direct map, namely $|\Psi(Z)|$ substitution $|\Psi(F(Z))|$.

Conclusions. The parameters proposed for the two S.S.C. options 2 and 2 and for the CERN L.H.C. seems to give us perfect stability form the beam-beam analysis made on the direct map (104 iterated and possibly extrapolated) and on the Birkhoff transform. Our analysis confirms the extrapolation method already proposed and used on ISR and previous proton-proton colliders (ISABELLE C.B.A....etc)(1) parameters.

References.
(1) M.Pusterla, G. Servizi, G. Turchetti "Non-linear dynamics....", Prooc. 12th Intern. Conf. on High Energy Accelerators... Fermilab (August 1983).