A method of injection into betatrons, which employs relativistic beams to overcome space charge limitations, is discussed. The method eliminates the need to modify the betatron with a toroidal magnetic field and avoids the attendant complications. Moreover, multiturn injection can reduce current and power requirements on the injector by orders of magnitude. Multiturn injection by rf linacs, however, may use a substantial fraction of the phase area in a betatron, which requires some means of painting the available phase space and missing the injector on subsequent turns. An example of such a betatron injection system is presented.

Introduction

The most important problem to be solved in scaling betatrons to high currents is injection. Betatrons have not been operated at vacuum currents above a few hundred amperes owing to space charge limitations at injection. In this paper we consider multiturn injection of relativistic electron beams by an rf linac. Injection of relativistic beams overcomes space-charge limitations at injection without a toroidal magnetic field and the complications a toroidal field introduces. Moreover, multiturn injection reduces injector power requirements by orders of magnitude. Phase painting techniques for this injection method are briefly described.

Many other betatron injection methods have been tried or proposed, including adiabatic variation of betatron frequencies, transient excitation of external coils, use of inductive transients or azimuthal nonuniformities, inflectorless injection, electrostatic inflection, use of corkscrew cathode, use of centrifugal force-cross-B drift, injection into plasma filled toroidal vessels, and injection into toroidal magnetic fields.

Injection of highly relativistic electron beams into betatron fields by using a low emittance electron linac to paint phase space both horizontally and vertically is a concept first proposed for an energy storage and power multiplication system. Further analysis and the first results of numerical simulations of this particular injection method are included in this paper with an accompanying simulation movie.

First Order Model of Electron Motion in Betatrons

To first order in \( \frac{x}{R} \) and \( \frac{y}{R} \), where \( x \) and \( y \) are horizontal and vertical displacements from the equilibrium orbit, and \( R \) is the radius of the equilibrium orbit, the magnetostatic betatron field is

\[
B_y = B_c (1 - sx/R), \\
B_x = -B_0 (sy/R),
\]

in which \( s \) is the betatron field index, \( 0 < s < 1 \). Using the two-mass approximation \( (v_z \gg v_x, v_y) \), the transverse components of the Lorentz force equation are

\[
\begin{align*}
my \left( \frac{dv_x}{dt} - \frac{v_x^2}{R + x} \right) &= -e \left( E_r - B_x B_0 \frac{y}{r} - B_x B_0 \right), \\
my \left( \frac{dv_y}{dt} - \frac{v_y^2}{R + y} \right) &= -e \left( E_r - B_x B_0 \frac{x}{r} - B_x B_0 \right).
\end{align*}
\]

Here \( e \) and \( m_y \) are the electron charge and relativistic mass, \( v_z = \beta \gamma c \) is the toroidal velocity component, \( r = \left( x^2 + y^2 \right)^{1/2} \) is the closest distance from the equilibrium orbit and \( E_r \) and \( B_0 \) are the self fields of a uniform, continuous, circular-cross-section beam of linear charge density \( \lambda \) and radius \( a \),

\[
E_r = \frac{2\lambda e r}{a^2}, \\
B_0 = B_x E_r.
\]

The equations reduce to

\[
\begin{align*}
X &= \left( \frac{\gamma^2}{c^2} (1 - s) - \frac{1}{s^2} \right) x = 0, \\
Y &= \left( \frac{\gamma^2}{c^2} - \frac{1}{s^2} \right) y = 0
\end{align*}
\]

where

\[
\frac{eB_0}{\gamma m_c} = \Omega = \frac{1}{\gamma} \left( \frac{4\pi e^2}{m_y} \right)^{1/2} \Omega_0 = \frac{\omega_0}{\gamma}.
\]

In this approximation, the electrons oscillate stably about toroidal equilibrium orbits as long as \( s \) and \( 1-s \) both exceed \( \frac{1}{s^2} \). This condition limits beam density to

\[
\eta_{\text{max}} = \frac{\gamma^3 m_c^2}{4\pi e^2 R^2}.
\]

The upper limit on beam density severely constrains injection of nonrelativistic electron beams into betatrons, but owing to the \( \gamma^3 \) factor, allows injection of intense relativistic beams.

Beam Emittance

For a highly relativistic beam well below the space charge limit in a simple betatron of index \( s \), the motion of electrons about the equilibrium orbit is

\[
X = x_0 \cos \left( (1-s) \frac{1}{s} x/R \right), \\
Y = y_0 \cos \left( s \frac{1}{s} x/R + \phi \right)
\]

**Permanent address:** Los Alamos National Laboratory, Los Alamos, New Mexico 87545.

**Permanent address:** U.C. Irvine, Irvine, California.

**Permanent address:** Mission Research Corporation, Albuquerque, New Mexico.
where \( x_0 \) and \( y_0 \) are amplitudes of the betatron oscillations, \( \phi \) is a constant phase, and \( z \) is distance along the equilibrium orbit.

In the horizontal plane, the area in phase space of an ellipse traced out by an electron having maximum displacement \( x_m \) is

\[
e_x = \pi x_m^2 = n(1-s) x_m^2 R.
\]

The emittance in the \( x \) dimension of a beam accelerated by an rf linac is given by the Lawson-Penner formula

\[
e_x = n K I^{2/3}(mA) \gamma^{-1} mm-mrad,
\]

where \( I \) is the average beam current in mA, and \( K \) is a constant typically in the range \( 2 \leq K \leq 8 \).

If a beam is injected into a betatron at a matched radius in the horizontal direction of \( \gamma_{x0} \), given by

\[
f_{bx} = \pi (1-s) \frac{r_{bx}^2}{R},
\]

it will neither expand nor contract inside the betatron. The corresponding matched radius in the vertical direction is given by \( e_{by} = \pi s \frac{r_{by}}{R} \). Thus, \( r_{bx} \) and \( r_{by} \) are the minimum radii that an elliptical beam can maintain upon injection into a betatron.

Painting Phase Space

The maximum number of turns that can be injected into a betatron by completely filling the available four dimensional phase space is

\[
N_{\text{max}} = \frac{b^2}{r_{bx}^2 r_{by}^2},
\]

where \( b \) is the minor radius of the betatron. Access to a substantial fraction of phase space can only be made available, however, if measures are taken to point phase space. Any method is also subject to the efficiency condition that a substantial fraction of the electrons miss the injector on subsequent turns.

We have considered a number of methods for painting phase space and missing the injector with the beam during multturn injection over many turns. All the methods had the following two features in common:

1) injection at the matched beam radius through a shielded injector port of comparable size protruding inside the betatron cavity, and

2) some method of sweeping the beam both horizontally and vertically to access all four dimensions of transverse phase space.

In Ref. 1 we suggested injecting the beam at the outer wall on the median plane and ramping the magnetic field during injection to bring the equilibrium radius from the outer wall to the toroidal axis. While the electrons are being adiabatically drawn radially inward, the beam is swept once vertically by an external dipole pair to access phase space in the vertical dimension and to keep the beam from returning to the injector site until the equilibrium orbit has moved inward by one injector port diameter. We have also considered variations of this method involving temporal and spatial changes of the betatron index.

Recently, we have settled on a method that appears to be the simplest, practicable, and most effective in terms of filling phase space and missing the injector. The method does not involve any external beam sweeping or any changing gradients or fast fields whatsoever. The injector is chosen to be in the upper or lower outside quadrant of the betatron minor cross section tangential to the wall of the betatron and at a point 45° from the median plane about the toroidal axis (for square or circular cross section betatrons). The betatron index is chosen to be

\[
s = \frac{1}{2} - \frac{1}{2K}\n
\]

where \( N \gg 1 \) is the number of turns to be injected.

Under these conditions, with no higher order effects, if a circular beam of radius \( r_b \) is injected into a betatron having circular cross section of radius \( b \), then the beam trajectory is given by

\[
x = \frac{h r_b}{2k} \cos \left( \frac{1}{2} + \frac{1}{2K} \right) \theta,
\]

\[
y = \frac{b r_b}{2k} \cos \left( \frac{1}{2} - \frac{1}{2K} \right) \theta.
\]

After \( N \) turns, the phase of the horizontal oscillations will have advanced \( 2\pi \) over the phase of the vertical oscillations, and for large \( N \) the phase area of the betatron will be nearly uniformly occupied. This is accomplished without any changes of gradients or fields internally or externally and with a fixed (unswep) beam. Aside from simplicity, the advantage that this method has over our earlier concept is that more of configuration space in both dimensions is available to the beam immediately upon injection, and the likelihood of collision with the injector is reduced.

As expected, particle simulations show that the beam qualitatively follows the analytic trajectory, but not closely enough that the beam can be confidently predicted to miss the injector on that basis alone. Instead, the betatron cross section may be chosen large enough so that the probability is high that a substantial fraction of the electrons miss the injector. If we assume a statistical loss on the injector of a fraction \( a^2/b^2 \) of the total number of circulating electrons during each turn, where \( a \) is the injector radius, then the fraction of injected electrons surviving after \( N \) turns is

\[
P_{\text{miss}}(N) = \frac{1}{N} \sum_{k=1}^{N} (1 - a^2/b^2)^k.
\]

For example, during 80 turns, \( P_{\text{miss}} = 68\% \) for \( b = 10a \) and \( P_{\text{miss}} > 90\% \) for \( b = 20a \).

In practice, \( P_{\text{miss}} \) will probably be selected to be 10-50%, and the field index of the betatron "tuned" to high efficiency injection of 80-96%. A lower \( P_{\text{miss}} \) allows a smaller minor cross section and less dilution of emittance. The emittance of the circulating beam after injection is complete, but before acceleration within the betatron has begun, is

\[
e_x = m b^2/2 R,
\]
an increase of $b'^2/r_b$ over the emittance of the linac beam. Adiabatic damping of betatron oscillations during acceleration from $\gamma_0$ to $\gamma$, however, will reduce the emittance of the circulating beam before extraction to

$$e_x \approx \left( \gamma_0/\gamma \right) n b'^2/2R.$$ 

Example

As an example we consider injection of 10 kA into a betatron having major radius $R = 250$ cm, minor radius $b = 10$ cm, and betatron field $B_0 = 544$ G. We assume linac beam parameters roughly corresponding to PHERMIX Upgrade, namely an average current $I_b = 125$ A, $\gamma = 80$, and an emittance $\epsilon_x = 7 \text{ mm-mrad}$. Then the matched beam radius inside the betatron is 0.5 cm. At least eighty turns are needed to store 10 kA, implying the pulse train must be at least 4.2 $\mu$s and the betatron index should be $n = 0.49$. Statistically, the efficiency of injection is expected to exceed 90%. After acceleration to, say, $\gamma = 1000$, the radius of the 10 kA beam inside the betatron will be about 2.8 cm, and the emittance about 200 nm-mrad. It should be possible to maintain high efficiency and achieve much lower emittance dilution by using a smaller minor radius and carefully tuning the field index.

Simulation Results

Preliminary results of the first two-dimensional particle simulations of our injection method using the fully self consistent, electromagnetic, relativistic particle simulation code IVORY are shown in an accompanying movie. The betatron and injection beam parameters used were the same as those given in the example above, except that for reasons of economy a continuous rather than bunched beam was injected, the trajectories were followed for 5.88 turns, and a square rather than circular betatron minor cross section was used.

Figure 1, Positions of each turn in square cross section of betatron at head of beam after 5.88 turns from IVORY simulation and simple analytic model. Numbers $n$ represent nth turn injected.

At 0.5 cm radius, the beam was matched to the betatron as expected, and maintained its radius through all turns. When the beam crossed itself on another turn, a breathing mode was introduced on the beam, but was of small amplitude, because of the weak space charge forces.

Conclusions

A simple method has been described for multiturn injection into a betatron by painting phase space vertically and horizontally with a highly relativistic electron beam from a linac. No pulsed fields of any kind are needed either. Multiturn injection reduces power requirements on the injector by orders of magnitude. Preliminary particle simulations have demonstrated well-behaved and well-understood injection of 750 A over six turns with no apparent loss of particles and no apparent limits to higher currents. Future analysis and simulations will determine effects of bunched beam injection, energy spread, and high circulating currents.

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References