A relevant application of the recent developments of non-linear dynamical theories lies in providing the description and the consequent calculation of the interactions among charged particles that either belong to a given beam or to different beams in collision with each other in storage rings.

Indeed, present and future generation accelerators need a detailed reliable simulation of their behaviour which takes into account the dominant non-linear effects to achieve a desired degree of efficiency and luminosity. To this end and adequate mathematical description of the beam-beam interaction, in both versions weak-strong and strong-strong, appears necessary primarily.

To this note we focus our attention on the weak strong case and particularly on the proton machines leaving the electron machines to a future investigation, because of the radiation damping which plays a central role in these systems. On the contrary in the proton-proton and proton-antiproton rings one may neglect this phenomenon, still acquiring a fairly realistic good simulation; more precisely we are going to work with dynamical conservative systems whereas the electron machines require the dissipative non-linear theory.

The motion of a test-particle around its main reference orbit is analysed by a conservative mapping that follows from periodical kicks, due to the forces of the beam-beam collisions, that solicitate the otherwise regular betatron oscillations. Indeed, present and future generation accelerators need a detailed reliable simulation of their behaviour which takes into account the dominant non-linear effects to achieve a desired degree of efficiency and luminosity. To this end and adequate mathematical description of the beam-beam interaction, in both versions weak-strong and strong-strong, appears necessary primarily.

The conservative mappings and Birkhoff method

We here introduce the mappings caused by kicks on the betatron oscillations in the interaction region for fixed angle and head-on collisions between two beams. If we call (as in usual conventions) the horizontal betatron elongation and the vertical one we may write, for head-on collisions between bunched beams, the following set:

\[
\begin{align*}
x_n &= x_{n-1} + S_x y_{n-1} - C_x x_{n-1} f(x_{n-1}, y_{n-1}) x \\
y_n &= y_{n-1} + S_y y_{n-1} - C_y y_{n-1} f(x_{n-1}, y_{n-1}) y
\end{align*}
\]  

where \( v_x, v_y \) are the betatron tunes; \( x_n, y_n \) the particle coordinates (in the motion relative to the reference orbit) at the instant of the \( n \)-th kick, \( x'_n, y'_n \) the left derivative versus the revolution angle \( \delta = \omega_R t + \delta_0 \) \( (\omega_R = 2\pi/T_R) \) = revolution frequency, just before the \( n \)-th kick (more exactly at \( \delta = 2\pi/m \) with \( m = 27/\delta \), \( m \) indicating the number of interactions of the test particle per revolution); \( C_x = \cos(v_x \delta) \), \( S_x = \sin(v_x \delta) \) and similarly for \( C_y \) and \( S_y \); \( f_x, f_y \) are the coupling constants of the beam-beam forces and \( \phi(x, y) \) is the function representing such forces. They all are uniquely defined by simple electromagnetic formulae in terms of the charge distribution in the "strong" beam that hits our test particle.

The mapping \( f(x, y) \) becomes one dimensional only for the vertical variable, in proton-proton rings (normally unbunched beams) such as ISR, ISABELLE, etc., again in absence of damping, noise and coupling to synchrotron oscillations. One readily shows that eq. (1) represent a conservative simplectic mapping because the instantaneous periodic kicks derive from the electromagnetic interaction theory which has a Hamiltonian formulation. If we assume a cylindrically symmetric distribution of charge in the dense beam, \( \phi(x, y) \) simplifies to a function of \( x^2+y^2 \); hence, a gaussian distribution we obtain the simple formula:

\[
\phi(r^2) = \frac{1}{2\pi} \exp\left(-\frac{x^2+y^2}{\sigma^2}\right)
\]

for any \( \phi(r^2) \) the linear mapping reads, for the \( x, x'/x \) canonical variables.

The Birkhoff procedure that we now intend to apply in a generalized version to the mapping (1), consists of searching for a formal simplectic transformation, for our case in \( R^2 \), that reduces the mapping to "normal form", namely the form of an integrable mapping. The first step is a linear transformation such that the linear mapping can be written as a standard product of rotations (transformation of a torus into itself). We may thus exclude those values of the parameters that make the origin linearly unstable.

Since \( \phi(r^2) = 1 + O(r^2) \) the linear mapping reads, for the \( x, x'/x \) canonical variables.

\[
\frac{dx}{dt} = C_x x + S_x y
\]
\[
\frac{dy}{dt} = C_y y + S_y y
\]
and similarly for $y$, $y'/\nu_y$ because the equations (1) decouple in the linear limit. 

Eq. (4) is a rotation of an angle $\alpha_i$ with
\[
\cos \alpha_i = \frac{C_x}{\sqrt{2}x} \quad \sin \alpha_i = \frac{S_x}{\sqrt{2}x} \quad \text{where} \quad \left| \frac{C_x}{\sqrt{2}x} \right| < 1
\]
provided that \(\left| \frac{C_x}{\sqrt{2}x} \right| < 1\).

In the complex variable $z$ representation with
\[
z = x + i \frac{y}{\sin \alpha_i} \quad \left( z = x + x' \right)
\]
eq(4) becomes simply $z = e^{i\alpha_i} z^\prime$ and similarly for
\[
(y, y'/\nu_y) \quad \text{with} \quad z^\prime = e^{i\alpha_2} z^\prime_2 - 1',
\]

The whole non-linear mapping (1) now takes the following representation:
\[
z^{n+1} = e^{i\Omega_k} \left( z^n + i\frac{\Omega_k}{k^n} \frac{z^n}{z^{n-2}} \right) \quad \text{where} \quad \Omega_k = \frac{\partial z}{\partial \alpha_k} \quad \text{with} \quad k = 1,2 \quad \text{and} \quad \delta \in \mathbb{C}, \xi \in \mathbb{C}, \nu \in \mathbb{C}_1, \nu \in \mathbb{C}_2
\]

\[
z^{n+1} = e^{i\Omega_k} \left( z^{n} + i\frac{\Omega_k}{k^n} \frac{z^n}{z^{n-2}} \right) \quad \text{with} \quad \Omega_k = \frac{\partial z}{\partial \alpha_k} \quad \text{with} \quad k = 1,2 \quad \text{and} \quad \delta \in \mathbb{C}, \xi \in \mathbb{C}, \nu \in \mathbb{C}_1, \nu \in \mathbb{C}_2
\]

It is easily possible to check that eq.(7) is symplectic if $z$ and $z'$ are taken as conjugate variables.

At this point we follow the Birkhoff method and seek a symplectic transformation:
\[
z = \phi_k(c_1, c_2, c_1', c_2')
\]
in such a way that the new mapping takes the form of a twist mapping:
\[
\zeta' = e^{i\Omega_k} \left( |c_1|^2, |c_2|^2 \right) \quad \zeta
\]
\[
(k = 1,2)
\]
where the functions $\Omega_k$ must verify the symplecticity constraints
\[
\{ \Omega_k^{(i)} \} \quad \{ |c_j|^2 \} = \{ \Omega_k^{(i)} \} \quad \{ |c_j|^2 \}
\]
for $k \neq j$.

$\phi_k$ and $\Omega_k$ are now expanded as series of homogeneous polynomials $\phi_k(\xi)$ and $\Omega_k(\xi)$ in the variables $\xi, \zeta, \eta$. In particular $\Omega_k(\Delta)$ must be considered a polynomial function of the square of the moduli $|\zeta|^2$:
\[
\phi_k = \frac{\partial}{\partial \alpha_k} \left( \zeta_1, \zeta_2, \eta_1, \eta_2 \right)
\]
\[
\Omega_k = \frac{\partial}{\partial \alpha_k} \left( |\zeta_1|^2, |\zeta_2|^2 \right)
\]

The coefficients of $\phi_k$ and $\Omega_k$ must be determined by solving recursively the functional equations:
\[
\phi_k(e^{i\Omega_1} c_1, e^{i\Omega_2} c_2, e^{-i\Omega_1} \eta_1, e^{-i\Omega_2} \eta_2) = e^{i\Omega_1} \phi_k
\]
\[
+ e^{i\Omega_2} \frac{\partial \phi_k}{\partial \eta_2} \left( \zeta_1, \zeta_2, \eta_1, \eta_2 \right) + e^{i\Omega_1} \frac{\partial \phi_k}{\partial \eta_1} \left( \zeta_1, \zeta_2, \eta_1, \eta_2 \right) - 1
\]

One readily sees that the functional equation determines the even order polynomials $\phi_k(2m)$ uniquely whereas the coefficients of $\Omega_k(2m)$ are evaluated together from identifying the odd order terms of eq.(10), with the exception of the coefficients of $\phi_k(2m)$ of the type $\zeta_k(1x)^{j(2m-j)}$ with that do not come out from the functional eq.(10) and must be computed in a different way. Indeed one can evaluate their real parts by imposing that the Poisson parentheses $\{\phi_k, \phi_m\}$ verify the symplectic property and consequently they have no powers of terms such as $\zeta_k(1x)^{j(2m-j)}$. This constructive approach must finally check the consistency conditions $\zeta_k, \zeta_m = 0$, $\{\phi_k, \phi_m\} = 0$.

At present the whole procedure has not yet completely been developed at all orders (one needs the expansion series of $\phi$), for the two-dimensional case, but a calculation performed up to the third order shows that we wanted consistencies do actually occur.

From the physical point of view the interest of this analysis stays in the possibility of passing from $\zeta, \zeta'$ to $\zeta, \zeta'$ and back, thus obtaining quite accurate extrapolations of the $n$ for very high values ($\approx 10^{10}, 10^{11}$).

The one-dimensional case

A complete development of this generalized Birkhoff-procedure and control of the error introduced by the truncation of the series have been achieved in the one-dimensional problem, this case after performing the linear transformation can be written as eq.(7) with $z = \pi$ and has the function $\phi$ defined by eq.(3). Its normal form follows from the transformations:
\[
z = \phi \left( \zeta, \zeta'^2 \right) = \zeta + \frac{\pi}{2} \frac{\Omega_k}{k^n} \phi_k(\zeta, \zeta'^2)
\]
\[
(k = 1,2)
\]
\[
z = \phi \left( \zeta, \zeta'^2 \right) = \zeta + \frac{\pi}{2} \frac{\Omega_k}{k^n} \phi_k(\zeta, \zeta'^2)
\]
\[
\text{and the formal } \phi \text{ inversion}
\]
\[
z - \psi \left( \zeta, \zeta'^2 \right) = \zeta - \frac{\pi}{2} \frac{\Omega_k}{k^n} \phi_k(\zeta, \zeta'^2)
\]
\[
\text{we then have:}
\]
\[
z' = e^{i\Omega_k(\zeta, \zeta'^2)}
\]
\[
\Omega_k = \frac{\partial}{\partial \alpha_k} \left( |\zeta|^2 \right)
\]

All these series are formal but uniquely defined except for $\text{Im} (a_n)$ which is redundant and can be chosen equal to $a_n$. The Poisson parentheses guarantee the symplecticity check completely and are equivalent to the Jacobian condition $\{ \Omega_k, \phi_k^* \} \approx 0$.

An automatic procedure for computing the $\psi, \phi, \psi$ series has been prepared and it allows a truncation to orders higher than $n=100$ for quadratic non-linearity. For non-linear functions $\phi$ that are given by series, as the case of the beam-beam mapping, one is forced to take lower order truncations which seem however sufficient for practical calculational purposes.

Furthermore, the divergent character of the Birkhoff series, carefully analysed for the quadratic mapping,
reveals general features that are independent of the
non-linearity introduced and most probably extensible
to the many-dimensional problem. One discovers $\chi$, looking at the ratios $\sigma_{2n} = (|\sigma_{2n}|/|\sigma_{2n+2}|)$,
\[
\lambda(n) = (|f_n(\epsilon^{10}_n, e^{10}_n)|/|f_{n+2}(\epsilon^{10}_n, e^{10}_n)|)
\]
for $n = 1, 2, \ldots$, that a sequence of pseudo-convergence
criteria $\delta_k$, $\delta_{k+1}$, $\delta_{k+2}$, monotonically tending to zero
$\delta_k$, the jet forms use to the next one related to
the closest resonances namely the rational that ap-
proximate $\pi/27$ in its continued fractions. Such a pic-
ture shows openly the asymptotic property of the Birk-
hoff series: one finds better and better agreement
between the formal and the true dynamics by sufficiently
increasing the truncation order of the series and
simultaneously restricting the domain of application
to smaller and smaller disks around the origin.

The asymptotic character of the approximation
of the twist mapping to the exact dynamics is seen im-
mediately if we compute the discrepancy
\[
\delta_n = |f_n(\epsilon^{10}_n, e^{10}_n) - z_n| \quad n \text{ being the truncation}
\]
order, $n$ the iteration order. The result is
\[
\delta_n \sim n(\epsilon^{10}_n/R)^{n+1}; \text{ the linear dependence of } n \text{ is ex-
}
plained by a systematic error $\delta R$ in the winding function
which can be computed and recovered to increase the
accuracy if one wishes to obtain extrapolations
of $z_n$ for very large $n$, including the phase information;
we recall that
\[
z_n = \delta(e^{10}_n, e^{-10}_n).
\]
Several types of information could be achieved indeed
from the $\delta$ function only, which specifies the invar-
iant curves completely: indeed, if $z_n$ belongs to an
invariant curve the minimum and maximum distance $R_{\text{max}}$
of its points from the origin may be obtained min-
in its analytical form:
\[
R_{\text{max}} = \min \left\{ |\epsilon^{10}_n + i c^{10}_n - a| \right\} \quad \left(16\right)
\]
\[
\max a < 2\pi \quad \text{with } c^{10}_n = \omega(z_0)^a_{10}
\]
Similarly one easily obtains the maximum value
of $|x|$ by taking $a = \min \text{Re}(|c^{10}_n| e^{10}_n, |c^{10}_n| e^{-10}_n)$.
The complete knowledge of $x_n$ after $n$ kicks, allows us
to evaluate the emittance
\[
\sqrt{2} \frac{1}{\lambda^2} \times x_n^2
\]
and more sophisticated physical quantities such as the polarization of the test particle (see next section).

Beam-beam effects on polarized particles

If the beam is made with polarized particles
 spontaneus polarization due to radiation in the
electron beams or polarization introduced on purpose
in proton, antiproton beams) the correct evaluation
of the depolarization due to non-linear dynamical col-
lisions becomes very hard and involves for very high
energy particles the Dirac spin $\epsilon$ equation $\delta$, which
connects directly the orbital motion with the spin
motion. One however easily sees that a semiclassical
description still may make sense under the physical
conditions that are normally verified for the test par-
ticle motion in the rings. We refer here to the Thomas-
BMT equation for the operator $S$ that represents the
spin operator in the rest frame of the particle to
which the intrinsic magnetic moment $\mu_0 = (g - 2)/2mc$ is
associated. In the lab. system $S$ obeys the equation
\[
dS/dt = c/m \left( a + \frac{1}{2} \right) S - a \gamma S \left( S^2 \right) = 
\]
\[
- \left( a + \frac{1}{2} \right) \gamma \times J + \frac{\hbar}{2mc} \nabla_S \cdot \nabla c
\]
\[
\left(17\right)
\]
where $a = (g - 2)/2$ (the so-called anomaly).
We separate the static field $\mu_0$ in the lab. system,
from the fields created by the "strong" beam in perio-
dic collisions with the test particle. If the period
of precession of $S$ around $\mu_0$ is large (as it is normal-
ly compared with the collision time, we can write
\[
\bar{S} = S_0 + \tilde{S}, \quad \delta_n(\theta - n \theta_{\text{m}})
\]
\[
\bar{S} = \tilde{S}, \quad \delta_n(\theta - n \theta_{\text{m}})
\]
\[
\beta_0 \text{ is static and should be uniform in an ideal ring but}
\text{we can take it as a space dependent function also; } \beta_1, \beta_2,
\text{ exhibit a functional dependence already considered}
\text{in the betatron motion equations.}
\]
\text{Eq.} (17) can be easily integrated giving the spin-
mapping equation in terms of the precession solution,
that holds in absence of "kicks", and in terms of the
betatron motion mapping; more specifically we have
\[
\bar{S} \left( \frac{2\pi}{m} \right) = \left[ 1 - \bar{R}_n \bar{x}^{-1} \right] \left[ 1 - \bar{R}_n \bar{x}^{-1} \right]
\]
\[
\tilde{S} \left( \frac{2\pi}{m} \right) = \tilde{S} \left( \frac{2\pi}{m} \right)
\]
\text{where the vector } \tilde{S}, \text{ is}
\[
\tilde{S}_n = (a + \frac{1}{2}) \tilde{R}_n \left( 2\pi \right) \tilde{x}_n \tilde{R}_n
\]
\[
\tilde{x}_n \text{ is essentially related with the complete elongation}
\text{of the betatron motion (easily obtained from the com-
}plex \tilde{z}_n \text{ and } \tilde{P}(\tilde{S}, \tilde{z}(2\pi/m)(k-1)) \text{ is the precess-
}ion solution of } \tilde{S}(0) \text{ in the interval } (2\pi/m)(k-1) \leq \tilde{C}(2\pi/m)k
\text{ in absence of beam-beam collisions. For an ideal}
uniform } \tilde{R} \text{ chosen along the vertical axis } \tilde{z}, \text{ the pre-
cession angular velocity is } \omega_0 = (e/\gamma c) \left( \tilde{R} / \tilde{R}_m \right)
\text{ and}
\[
\tilde{R}(0) \left( 2\pi \right) \tilde{R} \left( 2\pi \right)
\]
\[
\text{where } \tilde{R} \text{ is the rotation matrix:}
\]
\[
R(\tilde{S}) = \left| \begin{array}{ccc}
\cos(\tilde{v}_0) & \sin(\tilde{v}_0) & 0 \\
-\sin(\tilde{v}_0) & \cos(\tilde{v}_0) & 0 \\
0 & 0 & 1
\end{array} \right|
\]
\text{Numerical results}

A direct iteration of the mapping given by eq. (1)
was already considered in several papers $\delta$ with
the clear purpose of extrapolating to much later times
the information obtained from a convenient number of kicks
(order of magnitude $10^6$). This approach was essentia-
ly based on algorithms of numerical extrapolations
which can be easily programmed on a computer.

The Birkhoff procedure, on the contrary, empha-
sizes the analytical features of the mapping allowing
a much deeper control of the errors, the approximations
and the parameters domain.

We here compare the two approaches (Birkhoff and
extrapolation), that may be considered complement-
ary, for the proton-proton ISL.

Similarly, one can analyse other proton-proton
rings of future construction (ISABELLE, for instance).
We here report, for completeness, also unpublished, the
two-dimensional extrapolation calculations based on the mentioned extrapolation algorithms \((\varepsilon, \rho, \delta, \phi, r)\) for the pp CERN collider for which we intend to work out a Birkhoff analysis too.

**Birkhoff analysis**

a) \(v_y = 8.9\), \(\Delta v_y = -1.1 \times 10^{-3}\)
m (= number of intersections) = 8
(rotation angle of the linearized mapping) = 93.02°
Initial \(z\) value (cm) \(z = 0.127\), \(\text{Im} z = 0\)
\(\Re \zeta_0 = 0.12771\), \(\text{Im} \zeta_0 = -0.6317 \times 10^{-4}\)
\(P_{\text{min}} = 0.12758\), \(P_{\text{max}} = 0.1278\) (definition of \(P_{\text{min}}\) after 1000 iterations) in eq. (16).

After \(10^{10}\) kicks
\[
\sqrt{\frac{x_n^2 + \frac{1}{n^2} \Delta x_n^2}{n}} = 0.13705
\]

The modulus of discrepancy
\[
\left| \frac{\text{exact Birkhoff}}{\text{Birkhoff}} \right| = 1/n,
\]
\(n\) being the iteration number for which the direct mapping and the Birkhoff approach were compared, is constant once the truncation order \(N\) of the asymptotic Birkhoff series is established (in one case \(N = 15\)). We find for \(n = 1000\), \(n = 10^{-14}\) for the initial \(z\) value chosen. Consequently our iteration is remarkably good, via Birkhoff method up to \(10^{10}\), \(10^{11}\) iterations within an error of the order \(10^{-14}\), \(10^{-15}\).

b) Same machine parameters but initial \(z_0 = 0.2539\) give \(\Re \zeta_0 = 0.25485\), \(\text{Im} \zeta_0 = -0.504 \times 10^{-13}\)
\(P_{\text{min}} = 0.2539\), \(P_{\text{max}} = 0.2556\) (after 1000 iterations) at \(10^{10}\) iterations
\[
\sqrt{\frac{x_n^2 + \frac{1}{n^2} \Delta x_n^2}{n}} = 0.2460
\]
is reliable at least in the first figure.

c) If we increase \(z_0\) initial condition, keeping all other parameters identical, the Birkhoff method looses its reliability for truncation order \(N = 15\). For such a loss, in fact it appears that the truncated \(\phi(y)\), \((N=15)\) is more responsible than Birkhoff's truncation at large values of \(z_0\) (and consequently large initial elongations \(x_0\)).


The comparison of the result a) with the corresponding extrapolation for the limit \(n \to \infty\) of the sequence \(\frac{x_n + (1/n^2)x_{n+1}}{\sqrt{n}}\) obtained from the algorithms \((\varepsilon, \rho, \delta, \phi, r)\) is spectacularly good.

We finally give the extrapolation results for the pp CERN collider, using, as we said, the algorithm procedure (lim. \(n \to \infty\)):
\[v_x = 26.63; \quad v_y = 26.66; \quad \Delta v_x = 0.00283\]
\[\Delta v_y = 0.00288; \quad \nu_0 = 0.05 \text{ cm}; \quad x_0 = 0; \quad y_0 = 0.03 \text{ cm}\]
\(y_0 = 0\) (initial conditions).

On the basis of the function \(g(r^2)\) given by eq. (2) and on \(3 \times 10^6\) direct mapping eq. (1) iterations one obtains that the limits of \(x\) and \(y\) amplitudes may oscillate within a very narrow strip:

\[\text{min value of the limit for } x\text{-amplitude} = 0.049,\]
\[y\text{-amplitude} = 0.029\]

\[\text{max value of the limit for } x\text{-amplitude} = 0.051,\]
\[y\text{-amplitude} = 0.0308\]

The results are practically identical for 4, 6, 8 interactions per revolution and coincident in first 4 figures for all algorithms.

References