GENERAL DESIGN EQUATIONS FOR ISOCHRONOUS CYCLOTRONS
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II. General Design Equations for Orbit Properties

To obtain the general expressions for the betatron frequencies in the hard edge approximation, when $\varepsilon_1$, $\varepsilon_2$, $\eta_0$, $\mu_0$ and $\beta_0$ are functions of the radius, we may begin with the analytical expressions for the betatron frequencies in terms of the magnetic field coefficients (e.g., Smith and Garren)

$$
\nu_z = -\frac{\mu'}{2} + \frac{\pi}{2} \frac{a'' + b'}{n} + \frac{1}{2} \frac{1}{n^2} \frac{a'' + b'}{n^2}
$$

$$
= \frac{1}{2} \left[ 2 - \frac{\pi}{4} \frac{d}{dx} + \frac{d}{dx} \right] \frac{a'' + b'}{n^2}
$$

where

$$
\nu_z = 1 + \nu_z' + \frac{\pi}{2} \frac{n^2}{(n-1)(n-4)} \left( a'' + b' \right)^2
$$

$\mu'$ is already given by Eq.3. To obtain the radial derivatives of the Fourier coefficients $a'$ and $b'$ in the hard edge approximation, we Fourier analyse a step wave at radii $\rho_0$ and $\rho_0 + \xi_0$. With reference to Fig.3, the sectors will be displaced by $\Delta \phi_0$ and $\Delta \phi_0$, due to the spiralling at radius $\rho_0 + \xi_0$ (shown by the dotted wave) thus

$$
\rho_n(\rho_0 + \xi_0) - \rho_n(\rho_0) = N (B_h - B_v)
$$

and the cosine coefficient of the $n$th Fourier harmonic. Obtaining the corresponding expression for $q_n(\rho_0 + \xi_0) - q_n(\rho_0)$ and remembering that

$$
\tan \varepsilon = \rho_0 \frac{dn_0}{d\rho_0} + \frac{a'}{\beta_0} \frac{d\rho_0}{d\rho_0}
$$

and taking limits $\Delta \phi_0 \to 0$

$$
a'' + b' = \frac{N}{(n-1)(n-4)} \left( B_h - B_v \right) \left[ \frac{n^2}{(n-1)(n-4)} \right]
$$

Substituting from Eqs.3 and 8 in Eq.5 and retaining only the first three terms, gives the general hard edge expression for $\nu_z$

$$
\nu_z = -\frac{1}{n^2} \frac{2}{2\pi \mu_0 B_0} \left[ \Omega_0 + \frac{1}{2} \frac{\pi}{4} \frac{d}{dx} \right] \frac{a'' + b'}{n^2}
$$

where

$$
\Omega_0 = \frac{S_1 - S_3}{2} \tan \varepsilon_1 + S_2 \tan \varepsilon_2
$$

and

$$
\beta_0 = \frac{2}{\pi} \frac{B_0}{\mu_0}
$$

where

$$
S_1 = S_2 = \frac{1}{2} \frac{\pi}{4} \frac{d}{dx} \frac{a'' + b'}{n^2}
$$

The flutter $F^2$ and the time averaged field $\bar{B}$ over the orbit are given by

$$
F^2 = \frac{B_h - B_v}{B_0}, \quad \bar{B} = \frac{N_0}{(n-1)(n-4)} \Omega_0
$$

From Eqs. 3 and 6, the corresponding expression for $\nu_z$ becomes

"On the staff of the Nuclear Physics Division"
\[ \nu^2 = 1 + (\tan \eta - \tan \delta) / 4 \pm \nu_0 \pm [1 - \nu_0^2 \pm \nu_0 (2) \pm \nu_0 (3) / \nu_0 (4) \pm \nu_0 (5) / \nu_0 (6)] \]

omitting the third and other terms, which are only significant, because of the \( 1/N^2 \) dependence, when \( N \) is low.

Each equilibrium orbit in Fig.1 corresponds to a value of \( \nu \) given by

\[ \gamma (p_0) = \frac{(1 + p^2)^2}{(1 + p_0^2)^2} \]

where \( \eta \), the turning angle in the hill sector is related to \( q_0 \) by

\[ \cot \eta - q_0 = \cot \eta_0 - q_0 / \nu_0 \]

If \( \tau \) be the time period in an orbit corresponding to \( \nu \) at \( p_0 \), and \( \tau \) and \( \nu \) corresponding values for a reference orbit (e.g. the first orbit), then we have

\[ \frac{\tau}{\tau_1} = \frac{\gamma (p_0)}{\gamma (p_1)} \]

For isochronism, we require that the ratio \( \tau/\tau_1 \) be constant with the radius.

The Eqs. 9, 10 and 13 are valid for homogeneous field sectors and give the orbit properties directly in terms of the sector geometry. Eqs. 9 and 10 are valid for inhomogeneous field sector also. Since in this case, \( \phi_H \) and \( \phi_B \) will vary with the orbit angle (Fig.1), Eq. 13 may be used only as an approximation if \( B_H \) and \( B_B \) do not vary appreciably over the equilibrium orbit. For a separated sectored cyclotron, Eqs. 9, 10 and a modified form of Eq. 13 may be used with \( B_B = 0 \).

III. Scheme for arriving at the Optimum Sector shapes

The sectors alone cannot be made to provide the isochronous field for all the particles. They may however, be shaped to provide the isochronous field (i.e. \( \nu/\nu_0 \) constant) for a reference particle, i.e., in between the extreme cases such that the load on the trim coil currents is a minimum. Due to the different isochronous modulating fields under operating conditions, the \( (\nu, \nu_0) \) curves for each orbit will be different on the \( (\nu, \nu_0) \) graph. This family of lines for all the particles of interest must be kept away from a resonance region. Thus the optimum \( (\nu, \nu_0) \) curve exists for the reference particle which will keep the tunes of all other particles within the operating region on the \( (\nu, \nu_0) \) graph. Thus the optimum sector shapes must provide for the reference particle, simultaneously: i) a specific \( (\nu, \nu_0) \) curve on the \( (\nu, \nu_0) \) graph, and ii) \( \tau/\tau_1 \) constant with the radius.

Case of homogeneous field sectors

In the case of homogeneous field sectors, the optimum values of the parameters \( X \), \( B_H \) and \( B_B \) must be fixed. Also, \( \mu_B = \mu_B = 0 \). Since \( B_H \) and \( B_B \) are constant with the radius, an increase in the average field must be obtained by 'flaring' the sectors. The rate of flaring \( \delta n_0 / \delta p \) can be obtained through Eqs. 11, 12 and 13 for the condition \( \tau/\tau_1 \) constant and connected to the spiral angles \( \epsilon_1, \epsilon_2 \) through Eq. 11.

Further, assuming circular equilibrium orbits, we have approximately,

\[ \nu^2 = 1 + (p_0^2 - p_0) \]

where \( p_0 \) = cyclotron radius for the reference particle. Eqs. 14 can be used in conjunction with the required \( (\nu, \nu_0) \) curve for the reference particle to obtain the radial dependence of \( \nu_1, \nu_2 \). The Eq. 1 and 9 may now be solved simultaneously at several radii to obtain the \( \epsilon_1 (p_0) \) and \( \epsilon_2 (p_0) \) which will provide the \( \epsilon_1 (p_0) \) and \( \epsilon_2 (p_0) \) for the reference particle. If this is done, and making use of Eq. 7, the optimum sector contours are given by

\[ \theta_1 (p_0) = \int \left[ \frac{R_1^2 - 4 R_2^2}{2 R_3 (S_1 - S_2)} \right] \, dp_0 - \eta_0 (p_0) + \theta_0 \]

\[ \theta_2 (p_0) - \epsilon_1 (p_0) + \eta_0 (p_0) \]

where \( R_1 = \rho_0 \pm n_0 / \delta p_0 \), \( R_2 = (\beta_1 R_3^2 - \beta_0 R_4^2 - \beta_1 R_2^2) / \rho_0 \), \( R_3 = 1 / \beta_1 R_2^2 \), \( R_4 = 1 / \beta_0 R_2^2 \).

In practice, \( \delta \theta / \delta p \) may be calculated at a few radii, fitted to a polynomial in radius and integrated.

Case of inhomogeneous field sectors

In this case, the parameters \( B_H \) and \( B_B \) may also vary with the radius and \( \mu_B \neq 0 \). Thus the choice of the variables is not restricted to \( \epsilon_1 \) and \( \epsilon_2 \) only as in the case of the homogeneous field sectors. A scheme similar to that discussed for the homogeneous field case may be followed, depending upon the particular choice of the variables used.

Separated Sectored Cyclotrons

The above methods may be used, with \( B_B = 0, \mu_B = 0 \).

IV. Comparison of the hard edge equations with orbit integration results

A series of \( n_0^2 \) type spiral sectored electron cyclotron magnets were constructed in order to study the validity of the hard edge equations 9, 10 and 13 against orbit integration results for a wide range of parameters. The following three cases were considered: i) A three sectored, \( \beta = n_0^2 \) electron cyclotron magnet, ii) an eight sectored \( \beta = m_0^2 \) electron cyclotron magnet and iii) an \( n_0^2 \) sectored sectored cyclotron configuration.

Typical hard edge sector parameters obtained after machining and assembly on the pole pieces are shown in Table I. Magnetic field measurements were made in the median plane on a polar grid \( (r, \phi) \). \( B_0 \) and \( B_z \) refer to the measured maxima and minima at
each radius, $\mu_a$, $\mu_b$ and $\mu_c$ have been calculated using Eqs. 4a, 4b and 3 respectively.

The expected hard edge values of $\nu_a$, $\nu_b$ and $\nu_c$ for the three cases were obtained by substituting the sector parameters (e.g. Table I) in Eqs. 9, 10 and 13 respectively. The corresponding equilibrium orbit properties $\nu'_a$, $\nu'_b$ and $\nu'_c$ in the measured fields were computed with the equilibrium orbit code, ORBIT. A comparison of the hard edge $\nu_e$ and the corresponding orbit integration $\nu'_e$ is made for the three cases in Fig. 4a, b and c. A typical comparison of $\nu_e$ and $\nu'_e$ is made for $N=3$ in Fig. 4d. $\nu_e$ and $\nu'_e$ are compared for $N=8$, and for $N=8$ (separated sector) in Fig. 4e and f.

The hard edge values expected from Eqs. 9, 10 and 13 agree reasonably with the orbit integration results. Thus, Eqs. 15 and 16 may be used to obtain the preliminary optimum sector shapes. Once a scale model and orbit integration results become available, the exact optimum shape can be obtained by a differential corrective procedure using a modified Newton-Raphson method of successive approximations using the hard edge equations 9, 10 and 13.

For homogeneous field sectors, an alternative expression for $\nu_e$ when $\epsilon_1 \neq \epsilon_2$, has also been derived using the 'impulse approximation' approach of Richardson. For the special case of homogeneous field separated sectored cyclotrons, expressions corresponding to Eqs. 9, 10 and 13 derived by G. Schatz, using the matrix method, have been compared with the present set. These and other details will be available elsewhere.

References

1. L. Smith and A.A. Garren, UCRL-8598 (1959)
2. Orbit Dynamics and Design Theory for Optimization of Sector shapes in Isochronous Cyclotrons. A. Jain and A.S. Divatia, to be published.

Table I

<table>
<thead>
<tr>
<th>$N=3$</th>
<th>7.0 26.5 30.3 42.5 252.9 79.9</th>
<th>.01-.03</th>
<th>.24</th>
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<td>12.0</td>
<td>51.9 36.2 59.5 256.5 84.1</td>
<td>.09-.12</td>
<td>1.10</td>
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<td>$N=8$</td>
<td>8.0 12.2 35.9 42.2 257.8 94.5</td>
<td>.09-.15</td>
<td>.37</td>
</tr>
<tr>
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<td>.05-.83</td>
<td>1.10</td>
</tr>
<tr>
<td>8.0 11.0 16.5 44.6 55.0 287.4 97.3</td>
<td>.19-.60</td>
<td>1.09</td>
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For homogeneous field sectors, the dashed curves indicate values expected if the hard edge equations are used. The solid curves represent corresponding values obtained by orbit integration in the measured fields.