Analysis of a Compact Circular $TE_{0,1}$ - Rectangular $TE_{0,2}$ Waveguide Mode Converter *

Muralidhar Yeddulla †, Sami Tantawi , SLAC, Menlo Park, CA 94025, USA

Abstract

An analysis method for a three section mode transformer that converts a $TE_{0,1}$ circular waveguide mode to a $TE_{0,2}$ rectangular waveguide mode will be presented. Experimental results for this taper were earlier published in [1]. The middle section is a cylinder with a wall radius defined by $r_{wall} = a(1 + \epsilon \cos(2\theta))$, where $a$ is the radius of the circular guide and $\epsilon$ is a design parameter. This cylinder is connected on either side to a circular waveguide and a rectangular waveguide section respectively, through tapered waveguide sections. In this analysis we used a perturbation technique where the rectangular waveguide section’s wall radius is treated as a Fourier series expansion with $a$, the fundamental radius and $\epsilon$ the perturbation parameter. By applying the proper boundary conditions we optimize the taper dimensions to minimize conversion into spurious modes.

INTRODUCTION

In ultra high power RF systems such as those suggested for linear colliders, hundreds of megawatts of pulsed RF power is manipulated. Over-moded waveguides are widely used to increase the power handling capacity. Losses in the system are minimized by transporting power in circular waveguides in azimuthally symmetric modes such as the $TE_{0,1}$-mode. In many instances the RF power is easier to manipulate in rectangular waveguides than in circular waveguides [1]. Therefore, the power is often manipulated in rectangular waveguides and transported in circular waveguides.

In order to transport power between a circular waveguide and a rectangular waveguide, the two waveguides should be connected through a mode converter. It is possible to convert a $TE_{0,1}$-mode in a circular waveguide transitions into a $TE_{0,2}$-mode in a rectangular waveguide with a sufficiently smooth taper without scattering into other modes all along the taper. The length of the taper for such an adiabatic transition to a single mode at 11.424 GHz is about 18 inches which is excessive. We present a design method where the wave entering one end of the mode converter separates into two modes and recombines into a single mode while coming out at the other end, leading to a much shorter mode converter.

The idea behind this mode converter is as follows. Let a circular waveguide be transitioned to another waveguide of a certain cross section (let us call it an “oval” waveguide) through a linear taper (taper1) such that a $TE_{0,1}$-mode traveling through the circular waveguide would scatter into two modes, say $M_1$ and $M_2$ in the oval waveguide. Let a rectangular waveguide be transitioned to the same oval waveguide through another linear taper (taper2) such that a $TE_{2,0}$-mode traveling through the rectangular waveguide is scattered into the same modes $M_1$ and $M_2$ in the oval waveguide. Then, we may be able to achieve perfect mode conversion from a circular $TE_{0,1}$-mode to a rectangular $TE_{2,0}$-mode by transitioning through the three sections viz., taper1, oval waveguide and taper2, in that order and by optimizing the length of the three sections.

It is possible to design the mode converter using traditional numerical techniques like Finite Element Method (FEM). However, it would require a large amount of computational time to find an optimized solution for the design of the mode converter using FEM or other numerical techniques.

In this work we present a semi-analytical method to analyze a nonlinear waveguide whose cross section varies in two dimensions. We have used this method along with perturbation techniques to design the mode converter which needs much less computational time than FEM.

MODAL ANALYSIS

The wall radius of a waveguide that has a cross section with twofold symmetry ($r_w(\phi) = r_w(-\phi) = r_w(\pi + \phi) = r_w(\pi - \phi)$), may be expressed as,

$$r_w(\phi) = a_0 \left( 1 + \epsilon \sum_{p=1}^{P} \delta_p \cos(2p\phi) \right), \quad (1)$$

$\delta_p$ are Fourier expansion coefficients normalized to $\epsilon$ such that $\delta_1 = 1$. (If the waveguide wall cross section does not have twofold symmetry, then $\cos(2p\phi)$ should be replaced by $\cos[p\phi]$ in (1)).

For the oval waveguide, the cross section wall radius has a Fourier expansion only up to $P = 1$ which may be written as,

$$r_w,oval(\phi) = a(1 + \epsilon_{oval} \cos(2\phi)), \quad (2)$$

where $a$ is the radius of the circular waveguide and $\epsilon_{oval}$ is a small parameter which determines the deviation of the second section from a circular waveguide. The linear taper between the circular waveguide and the oval waveguide, taper1, also has a wall radius of the form of equation (2).

For a rectangular waveguide with side lengths $c$ and $d$, the wall radius is given by

$$r_{w,rect}(\phi) = \begin{cases} \frac{d}{2 \cos \phi}, & 0 < \phi < \tan^{-1}\left(\frac{c}{d}\right) \\ \frac{c}{2 \sin \phi}, & \tan^{-1}\left(\frac{c}{d}\right) < \phi < \frac{\pi}{2} \end{cases} \quad (3)$$

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The Fourier expansion coefficients in (1) for a rectangular waveguide can be expressed as Fourier integrals of the wall radius given by (3). For the case of taper2, the expansion coefficients in (1) can be linearly interpolated in $z$ in terms of the expansion coefficients of a rectangular waveguide. Thus, the mode converter’s Fourier expansion coefficients in (1) may be determined anywhere along the axis in terms of the design parameters, viz., $a_i$, $\epsilon_{oval}$, and the dimensions of the rectangular waveguide.

For TE modes, the mode vector function $\vec{e}_i$, which is proportional to the RF electric field inside the waveguide is given by [2],

$$\vec{e}_i = \hat{r} \times \nabla_\perp \Psi_i,$$  

(4)

$\nabla_\perp$ is the gradient operator transverse to the waveguide axis and $\hat{r}$ is the unit vector in the direction of the waveguide axis and $\Psi_i$ is the mode eigen function.

The eigen function for a mode in a nonlinear waveguide may be written as a Fourier series of the mode in a circular waveguide perturbed by a small expansion factor $\epsilon$, which may be expressed as

$$\Psi_s = \sum_i H \alpha_{2i} J_{2i} (k_{\perp s} r) \cos (2i \phi),$$  

(5)

where $\alpha_{2i} = \sum_j A_{2i,j} e^j$ and $k_{\perp s} = \sum_j \chi_j e^j$. $A_{2i,j}$ and $\chi_j$ are the expansion coefficients of the mode amplitude and mode cutoff wave number at any given cross section, respectively. We use subscript $s$ to represent any mode inside the mode converter.

By applying the boundary condition that the tangential component of the electric field at the waveguide wall is zero,

$$\vec{e}_s \cdot \frac{\partial \hat{r}_w}{\partial \phi} = 0,$$  

(6)

where $\hat{r}_w = r_w \hat{r}$, we can determine all the expansion coefficients $A_{2i,j}$ and $\chi_j$ in (5) at any cross section along the axis of the nonlinear waveguide and hence determine the eigen function for both modes.

**MODE COUPLING**

This section describes a method developed by Solymar [3] to estimate the scattering of modes in a nonlinear waveguide.

The inter-mode coupling in a nonlinear waveguide may be accounted through Telegrapher’s equations of the form,

$$\frac{dV_i}{dz} = -j k_{z_i} Z_i I_i + \sum_m T_{im} V_m,$$

$$\frac{dI_i}{dz} = -j k_{z_i} V_i Z_i - \sum_m T_{mi} I_m,$$  

(7)

where $V_i$ and $I_i$ are the mode voltage and current, $k_{z_i} = \sqrt{k^2 - k_{z_i}^2}$ is the uncoupled propagation constant for the $i$th mode, $k_{z_i}$ is the cutoff wave number of the $i$th mode, $k$ is the propagation constant in free space, $m$ denotes all other modes including the main mode in the waveguide, and $Z_i$ is the mode wave impedance.

The mode coupling coefficients given in (7) are given by,

$$T_{mi} = \int_S \hat{e}_m \cdot \partial \hat{e}_i \, dS,$$  

(8)

where $S$ is the cross sectional surface of the nonlinear waveguide.

Assuming that the modes considered are above cutoff, the mode voltage $V_i$ and mode current $I_i$ may be expressed in terms of forward and backward wave amplitudes, $A_i^+$ and $A_i^-$. We assume that only two modes, $M_1$ and $M_2$ are present inside the non-linear waveguide and there are no reflections. Under these conditions the amplitude of the $i$th mode due to the coupling with the $m$th mode is described by,

$$\frac{dA_i^+}{dz} + j k_{z_i} A_i^+ = S_{im}^+ A_m^+.$$  

(9)

where

$$S_{im}^+ = \frac{1}{2} \left[ \sqrt{\frac{k_{z_i}}{k_{z_m}}} T_{mi} - \sqrt{\frac{k_{z_m}}{k_{z_i}}} T_{im} \right],$$  

(10)

is the transfer coefficient between the two modes.

**RESULTS**

Inside the mode converter the modes $M_1$ and $M_2$ may be considered as perturbations of circular waveguide $TE_{0,1}$-mode and $TE_{2,1}$-mode respectively. Then the known expansion coefficients in (5) for mode $M_1$ are $A_{0,0} = 1$, $A_{0,j} = 0$ for $j \neq 0$, $A_{2i,0} = 0$ for $i \neq 0$ and $\chi_0 = 3.832$ (eigen number corresponding to $TE_{0,1}$-mode in a circular waveguide) and for mode $M_2$ are $A_{0,0} = 0$, $A_{2,0} = 1$, $A_{2,j} = 0$ for $j \neq 0$, $A_{2i,0} = 0$ for $i \neq 1$ and $\chi_0 = 3.054$ (eigen number corresponding to $TE_{2,1}$-mode in a circular waveguide). Using these known expansion coefficients the unknown coefficients $A_{2i,j}$ and $\chi_j$ may be determined by expanding the left hand side of (6) in $\epsilon$ and equating the expansion coefficients to zero to find the eigen functions for the two modes $M_1$ and $M_2$ anywhere inside the mode converter. The accuracy of the solution increases as we consider more number of RF harmonics, $H$, as well as more number of waveguide wall radius expansion harmonics, $P$, to represent a rectangular waveguide in (1). As $H$ and $P$ are increased the number of expressions that need to be solved to determine the expansion coefficients increases rapidly into thousands. This necessitates the use of a symbolic solver like Mathematica [4] which we have used in our calculations.

We have considered the rectangular waveguide wall radius as a Fourier expansion given in (1) with $P = 5$. Also, we have assumed that there are, $H = 6$, RF harmonics inside the mode converter. We have optimized the length of taper1, taper2 and oval waveguide as well as $\epsilon_{oval}$ to obtain the lowest level of the mode $M_2$ at the end of the mode.
converter at 11.424 GHz. The following are the dimensions of the mode converter that were obtained by this exercise. Radius of circular waveguide, \( a = 1.905 \text{ cm} \). Rectangular waveguide sides, \( c = 1.52 \text{ cm}, d = 1.82 \text{ cm} \). Taper length between circular and oval waveguides (taper1), \( L_1 = 2.43 \text{ cm} \). Length of oval waveguide = 1.975 cm. Taper length between oval and rectangular waveguide (taper2) = 3.0 cm. \( \epsilon_{\text{oval}} = 0.1132 \).

Figure 1: Level of mode \( M_2 \) at the end of taper1 calculated using HFSS and perturbation techniques.

We have studied the frequency response of the above geometry with different number of RF harmonics \( H \). We first consider only taper1 for our study. Fig.1 shows the normalized amplitude of the mode \( M_2 \) (normalized to the amplitude of the main mode \( M_1 \)) at the end of taper1 for \( H = 5 - 9 \), along with the results of HFSS field solver simulation (see reference [5]) for the same geometry as a function of frequency. We see that the frequency response has similar characteristics for all values of \( H \) except for a difference in the level of \( M_2 \). It is interesting to note that the level of \( M_2 \) changes very little for \( H = 5,6 \) and for \( H = 7,8 \). We also see from Fig.1 that the results of perturbation theory matches reasonably closely with that of HFSS field solver simulations at the end of taper1 (within 0.2 dB over the frequency range considered).

CONCLUSIONS

We have developed an analytical perturbation technique that was used to design a compact circular to rectangular waveguide transition. This technique predicts the level of a spurious mode inside the mode converter which is a nonlinear waveguide within reasonable accuracy compared to HFSS field solver simulations. As the perturbation technique presented in this work leads to an analytical solution, the calculations are much faster than HFSS. Therefore this technique, which is quite general, can be an attractive tool in the design and analysis of a wide variety of nonlinear waveguides.

REFERENCES