COUPLING IMPEDANCES FOR CORRUGATED BEAM PIPES
FROM IMPEDANCE BOUNDARY CONDITIONS

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Abstract
An equivalent wall impedance describing the electro-
magnetic boundary conditions at corrugated pipe walls is
introduced in the context of a general perturbative
approach for computing the longitudinal and transverse beam
coupling impedances in complex heterogeneous pipes.

INTRODUCTION
Coupling impedances are a powerful tool for studying
the interaction between a charged particle beam and the
surrounding chamber. Unfortunately, coupling impedances
can be usually computed only by numerical methods lead-
ing to computationally intensive design optimization pro-
cedures.

The combined occurrence of complex geometrical fea-
tures and/or the use of several different wall materials,
make the electromagnetic boundary value problem analyt-
ically almost untractable. As a matter of fact, only a few
analytic solutions for coupling impedances are available,
for simple cases where, e.g., the Laplacian is separable in
the pipe cross-section coordinates, and the boundary con-
ditions are very simple too (e.g., perfect conductors).

In this paper we estimate the longitudinal and transverse
coupling impedances for a pipe with corrugated walls us-
ing the general framework presented in [1] and summarized
below, using an impedance boundary condition (b.c.) of the
Leontóvich type, to account for the corrugations. An appli-
cation to a candidate LHC geometry is included.

COUPLING IMPEDANCES
IN COMPLEX PIPES
According to [1] the longitudinal and transverse beam
coupling impedances $Z_{||}(\omega)$ and $Z_{\bot}(\omega)$ of a simple,
unperturbed pipe (e.g., circular, perfectly conducting) as-
sumed known, can be related to those $Z_{||}(\omega)$, $Z_{\bot}(\omega)$ of an-
other pipe differing from the former by some perturbation
in the boundary geometry and/or constitutive properties, as
follows (beam at $\vec{r} = 0$) 1:

$$Z_{||}(\omega) - Z_{||}(\omega) = \frac{\epsilon_0}{\beta_0 c Q^2} \left\{ Y_0 \oslash \int_{\partial S} Z_{\text{wall}} E_{0n}^{(\text{irr.})}(\vec{r}) \right\} \otimes$$
$$\otimes \nabla_{\vec{r}_1} \left[ \beta_0 E_{n}^{(\text{irr.})}(\vec{r}, \vec{r}_1) + \beta_0^{-1} E_{n}^{(\text{sol.})}(\vec{r}, \vec{r}_1) \right] d\ell +$$
$$- \oint_{\partial S} \nabla_{\vec{r}_1} E_{0z}^{*}(\vec{r}, \vec{r}_1) \otimes \nabla_{\vec{r}_1} E_{n}^{(\text{irr.})}(\vec{r}, \vec{r}_1) d\ell \right\}_{\vec{r}=\vec{r}_1=0},$$

where $c = (\epsilon_0 / \mu_0)^{-1/2}$ is the speed of light in vacuum,
$Y_0 = (\epsilon_0 / \mu_0)^{1/2}$ is the vacuum characteristic admittance,
$\epsilon_0$ and $\mu_0$ being the vacuum permittivity and permeabil-
ity, $\beta_0$ is the relativistic factor, $Q$ is the total beam charge,$E_{n}^{(\text{irr.})}$, $E_{n}^{(\text{sol.})}$ are the solenoidal and irrotational parts of
the electric field, a suffix “0” identifies the unperturbed
quantities, and an impedance (Leontóvich) boundary con-
dition is assumed to hold at the (perturbed) pipe wall $\partial S$:

$$\hat{n}_n \times \left( \hat{n}_n \times \vec{E} - Z_{\text{wall}} \vec{H} \right)_{\partial S} = 0,$$

where $Z_{\text{wall}}$ is the pipe-wall complex characteristic
impedance and $\hat{n}_n$ is the unit vector normal to $\partial S$.

The first integral term on the r.h.s. of (1) and (2) is
nonzero if and only if $Z_{\text{wall}}$ is not identically zero on $\partial S$,
and accounts for the effect of the (complex) wall conduc-
tivity. The second integral term on the r.h.s. of (1) and (2),
on the other hand, accounts for the effect of the geometrical
perturbation of the boundary, and is non-zero if and only if
the unperturbed axial field component $E_{0z}$ is not identi-
cally zero on $\partial S$. Letting $\vec{E}_0$ in place of $\vec{E}$ in (1) and (2),
one obtains a first order perturbative formula for the beam
coupling impedances in the perturbed pipe.

CORRUGATED BEAM PIPES
Let

$$\vec{r} = \vec{r}_b(\theta) = \vec{R}_b(\theta) + \delta \vec{R}(\theta),$$

the (transverse) position of a point on the (perturbed) pipe
boundary $\partial S$, where $\vec{R}_b(\theta)$ defines the unperturbed bound-
ary $\partial S_0$, $\delta \vec{R}(\theta)$ describes the $z$-independent roughness,
and $\theta$ is the polar angle. To first order in the corrup-
gations,

$$E_{0z}^*(\vec{r}_b) \sim E_{0z}^*(\vec{R}_b) + \nabla E_{0z}^*(\vec{R}_b) \cdot \delta \vec{R}. \quad (5)$$

The first term in (5) is obviously zero (the unperturbed boundary is by assumption a perfect conductor). The un-
perturbed longitudinal field is related to the potential $\Phi_0$,

$$E_{0z}^* = -jk(1 - \beta_0^2) \Phi_0,$$
whereby
\[ \vec{E}_0^* = -\nabla \Phi_0^*. \] (7)
so that (5) becomes
\[ E_{0z}(\vec{r}_0) \sim jk(1 - \beta_0^2)E_{0z}(\vec{R}_0)\hat{n}_0(\theta) \cdot \delta \vec{R}_0. \] (8)

since the tangential component of \( \vec{E}_0 \) at the unperturbed boundary (perfect conductor) is zero.

Accordingly, the integral in (1) which accounts for the effects of the geometrical perturbation of the pipe boundary can be written, to first order:
\[ I_{\parallel} = -jk(1 - \beta_0^2) \int_{\delta S_0} \hat{n}_n(\ell) \cdot \delta \vec{R}_0(\ell) \cdot |E_{0z}(\ell)|^2 \, d\ell \] (9)
where \( \ell \) is a curvilinear coordinate on \( \delta S_0 \).

Similarly, to first order in the corrugation term \( \delta \vec{R}_0 \),
\[ \nabla_{\vec{R}_0} E_{0z}(\vec{r}_0) \otimes \nabla_{\vec{R}_1} E_{0z}(\vec{r}_1) \approx jk(1 - \beta_0^2)\delta \vec{R}_0 \cdot \hat{n}_0(\theta) \]
\[ \cdot \left[ \nabla_{\vec{R}_0} E_{0z}(\vec{R}_0, \vec{r}_0) \otimes \nabla_{\vec{R}_1} E_{0z}(\vec{R}_1, \vec{r}_1) \right]. \] (10)

Accordingly, the integral in (2) which accounts for the effects of the geometrical perturbation of the pipe boundary can be written, to first order:
\[ I_{\perp} = -jk(1 - \beta_0^2) \int_{\delta S_0} \hat{n}_n(\ell) \cdot \delta \vec{R}_0(\ell) \]
\[ \cdot \left\{ \nabla_{\vec{R}_0} E_{0z}(\ell, \vec{r}_0) \otimes \nabla_{\vec{R}_1} E_{0z}(\ell, \vec{r}_1) \right\}_{\vec{r}_0=\vec{r}_0=0} \, d\ell \] (11)

Comparison of (9), (11) to (1) and (2) shows that the roughness \( \delta \vec{R}_0(\theta) \) is "equivalent" to a non-uniform, purely reactive impedance loading
\[ Z_{\text{wall}}^{(\text{equiv.})} = -jk(1 - \beta_0^2)Z_0\hat{n}_0(\theta) \cdot \delta \vec{R}_0(\theta), \] (12)
laid down on the unperturbed pipe wall. It is also seen that, for the special case where \( \delta \vec{R}_0(\theta) \) is a random process, its statistical moments are simply related to those of the equivalent wall-impedance (12). These findings are more or less obviously related to the general formalism developed in [3] for describing (weakly) irregular surfaces in terms of impedance boundary conditions.

**CORRUGATED CIRCULAR PIPE**

As a simplest example, we refer to a corrugated perfectly conducting circular pipe. The unperturbed geometry is a smooth perfectly conducting pipe of radius \( R \). The unperturbed field produced at \( \vec{r} \) by a beam at \( \vec{r}_0 \) is
\[ \vec{E}_0(\vec{r}, \vec{r}_0) = \frac{Q}{2\pi \epsilon_0} \left\{ \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^2} - \frac{\vec{r} - \vec{r}_0(R/r_0)^2}{|\vec{r} - \vec{r}_0(R/r_0)^2|^2} \right\}, \] (13)

From (13) one readily obtains
\[ \vec{E}_0(\vec{r}) = \frac{Q}{2\pi \epsilon_0} \frac{\vec{r}}{|\vec{r}|^2} \] (14)
and:
\[ \lim_{r_0 \to 0} \nabla_{\vec{r}_0} \vec{E}_0(\vec{r}, \vec{r}_0) = \lim_{r_1 \to 0} \nabla_{\vec{r}_1} \vec{E}_0(\vec{r}, \vec{r}_1) = \frac{Q}{\pi \epsilon_0 r^3}. \] (15)

Now consider the perturbed case of a circular pipe with uniform wall impedance \( Z_{\text{wall}} \). Using eqs. (1) and (2) with \( \vec{E} = \vec{E}_0 \) together with (14) and (15), one readily obtains
\[ Z_{\parallel} = \frac{Z_{\text{wall}}}{2\pi R}, \quad \tilde{Z}_{\perp} = \frac{Z_{\text{wall}}}{\pi k_0 R^3} (\hat{u}_x \hat{u}_x + \hat{u}_y \hat{u}_y), \] (16)
in agreement with the known exact result [2]. One is therefore led to guess that eqs. (9), (11) should be likewise accurate for computing the coupling impedances contributed by corrugations. Hence, for a perfectly conducting pipe
\[ Z_{\parallel} = \frac{\langle Z_{\text{wall}}^{(\text{equiv.})} \rangle}{2\pi R}, \quad \tilde{Z}_{\perp} = \frac{\langle Z_{\text{wall}}^{(\text{equiv.})} \rangle}{\pi k_0 R^3} (\hat{u}_x \hat{u}_x + \hat{u}_y \hat{u}_y), \] (17)
where
\[ \langle Z_{\text{wall}}^{(\text{equiv.})} \rangle = -jk(1 - \beta_0^2)\frac{Z_0}{\delta S_0} \int_{\delta S_0} \hat{r} \cdot \delta \vec{R}_0(\ell) \, d\ell \] (18)
is the circumferential average of (12). It is seen that suitable (\( z \)-independent) corrugations can be used to compensate the remaining reactive terms in the beam coupling impedance at a specific frequency.

**LHC IMPEDANCE BUDGET**

The candidate LHC geometry includes two corrugated sections as shown in Fig.1 below. The corrugations con-

![Figure 1: A simplified candidate LHC geometry.](image)

contributions to the (reactive) impedance budget can be easily computed using (13) for the unperturbed field, with \( R \approx 14 \text{mm} \). Assuming \( \beta \approx 0.987 \), it is found from (18) and Fig.1 that \( \langle X_{\text{wall}}^{(\text{equiv.})} \rangle \approx -1.467 \times 10^{-8} \text{ohm at the beam circulation frequency (} \approx 11 \text{KHz).} \)
REFERENCES

