INVESTIGATION OF THE FLAT-BEAM MODEL OF THE BEAM-BEAM INTERACTION

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Abstract

At the interaction point of a storage ring collider each beam is subject to perturbations due to the electromagnetic field of the counter-rotating beam. For flat beams, a well known approximation models the beam by a current sheet which is uniform in the horizontal plane, restricting the particle motion to the vertical direction. In this classical model a water-bag beam distribution is used to find work-...
SOLVING THE EQUATIONS OF MOTION

We expand the linearized version of eqn. 9 using the ansatz

\[ f(J, \phi, s) = \sum_{n' l'} g_{n' l'}(s) e^{-\frac{J}{\epsilon} L_{n'}} e^{i l' \phi} \]

where the \( n' \)-th Laguerre polynomial is denoted by \( L_{n'} \) and the summation runs from 0 to \( \infty \) for \( n' \) and from \( -\infty \) to \( \infty \) for \( l' \). Substituting eqn. 10 and eqn. 11 into eqn. 9 we obtain eqn. 12

COHERENT BEAM-BEAM INSTABILITY

We solve the ODE 12 and rewrite the solution in matrix form such that the beam transport after one turn is described by a matrix \( T \) which acts on a column vector \( G \) that contains all \( g_{n l} \), i.e. \( G(C) = TG(0) \) and parametrize the beam-current by introducing the tune shift parameter

\[ \Delta \nu \equiv \frac{N r_e}{\gamma} \sqrt{\frac{2 \beta^*}{\pi \epsilon}}, \]

where \( \beta^* \) denotes the beta function at the interaction point. In order to decide whether the system is stable or not we have to find out what happens to an arbitrary initial perturbation after a large number of turns, i.e. one needs to consider the limit \( T^N \) where \( N \rightarrow \infty \). Every matrix norm of the latter quantity tends to infinity if the absolute value of all eigenvalues of \( T \) are bigger than 1.

RESULTS AND DISCUSSION

In Fig. 1 and 2 we have drawn a point if the absolute value of all eigenvalues of \( T \) is smaller or equal 1 for both the \( \sigma \)- and the \( \pi \)-mode. The first and second order resonances can be recognized clearly. Resonances of orders higher than 2 cannot be expected in our linearized model. From the diagrams we conclude that the inclusion of radial modes tends to stabilize the beam. In Fig. 3 and 4 we determine which mode becomes unstable by selecting the biggest component of the eigenvector which is associated with the largest eigenvalue. The plot shows that in the absence of dynamics in the radial direction \( l = \pm1 \) modes, medium grey and dark grey points indicate unstable \( l = \pm2 \) modes. The following modes were included: \( n = 0, l = -2 \ldots 2 \)

In Fig. 5 we computed the phase of the largest eigenvalue of \( l = \pm2 \) instabilities (\( \sigma \)-mode only) versus the perturbed tune. Light grey points indicate unstable \( l = \pm1 \) modes, medium grey and dark grey points indicate unstable \( l = \pm2 \) modes. The following modes were included: \( n = 0, l = -2 \ldots 2 \)
\[
\frac{\partial g_{nl}}{\partial s} + \frac{il}{\beta} g_{nl} \pm \frac{(2n)!\sqrt{\pi}}{2(2n-1)(2n+1)!} \frac{4\pi N_{\epsilon} \sqrt{2\beta\epsilon}}{2\pi} \frac{1}{2i} \left( \delta_{l,1} - \delta_{l,-1} \right) \delta_p(s) \left[ -4\epsilon \sum_{l'} g_{0,2l'+1} \frac{(-1)^{l'}}{2^{l'+1}} \right] \\
\pm \frac{2\beta(-1)}{2\pi} \frac{1}{4i} \frac{4\pi N_{\epsilon}}{\gamma\epsilon} \left( \delta_{n,0} - \delta_{n,1} \right) \left( \delta_{l,2} - \delta_{l,-2} \right) \delta_p(s) \left[ 2\sqrt{\frac{2\epsilon}{\beta}} \sum_{n'n'} g_{n'n',2l'} (-1)^{l'} \frac{(2n')!\sqrt{\pi}}{(2^{n'n'}n')!^2} \right] = 0
\]

POSSIBLE EXTENSIONS

We have extended our model to account for damping by synchrotron radiation. In order to obtain the equilibrium distribution 10 quantum excitation must be included as well. This turns eqn. 3 into a Fokker-Planck equation. In our preliminary computations we found that the graphs we presented above remain unchanged for realistic values of the damping and excitation coefficients. To simplify the Fokker-Planck equation we averaged over the phases in the damping and excitation terms but not in the beam-beam interaction term. This can be justified since the betatron phases in the terms for damping and quantum excitation change during one turn while the phase in the interaction term changes only once per turn.

Higher order resonances can be studied by not linearizing the integral in eqn. 9 and assuming that \( f_\pm \) contains only one radial mode. However, this procedure is complicated by the fact that eqn. 10 is not an equilibrium anymore. Ignoring these problems one can obtain plots similar to Fig. 1 with higher order resonances. For accelerators with different tunes for the rotating and counter-rotating beam a bigger transfer matrix which describes the evolution of the \( g_{nl} \) for both beams can be derived easily.

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REFERENCES