KINETIC STUDIES OF TEMPERATURE ANISOTROPY INSTABILITY IN INTENSE CHARGED PARTICLE BEAMS *

Edward A. Startsev, Ronald C. Davidson and Hong Qin
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08543

Abstract

This paper extends previous analytical and numerical studies [E. A. Startsev, R. C. Davidson and H. Qin, Phys. Plasmas 9, 3138 (2002)] of the stability properties of intense nonneutral charged particle beams with large temperature anisotropy ($T_{\perp b} \gg T_{\parallel b}$) to allow for non-axisymmetric perturbations with $\partial/\partial \theta \neq 0$. The most unstable modes are identified, and their eigenfrequencies and radial mode structure are determined.

LINEAR STABILITY THEORY

It is well known that in neutral plasmas with strongly anisotropic distributions ($T_{\perp b}/T_{\parallel b} \ll 1$) a collective Harris instability [1] may develop if there is sufficient coupling between the transverse and longitudinal degrees of freedom. Such anisotropies develop naturally in accelerators. For particles with charge $q$ accelerated by a voltage $V$, the longitudinal temperature decreases according to $T_{\parallel b} = T_{\parallel b}^0/2qV$ (for a nonrelativistic beam). At the same time, the transverse temperature may increase due to nonlinearities in the applied and self-field forces, nonstationary beam profiles, and beam mismatch. These processes may provide the free energy to drive collective instabilities, and lead to a deterioration of beam quality. The instability may also result in an increase in the longitudinal velocity spread, which will make the focusing of the beam difficult, and may impose a limit on the minimum spot size achievable in heavy ion fusion experiments.

We briefly outline here a simple derivation [2] of the Harris-like instability in intense particle beams for electrostatic perturbations about the thermal equilibrium distribution with temperature anisotropy ($T_{\perp b} > T_{\parallel b}$) described in the beam frame by the self-consistent axisymmetric Vlasov equilibrium [3]

$$f^0_b(r, p) = \frac{n_b}{(2\pi m_b)^{3/2} T_{\parallel b}^{1/2}} \exp\left(-\frac{H_{\parallel}}{T_{\parallel b}} - \frac{H_{\perp}}{T_{\perp b}}\right).$$

Here, $H_{\parallel} = p_\parallel^2/2m_b$, $H_{\perp} = p_\perp^2/2m_b + (1/2)m_b \omega_f^2 r^2 + e_b \phi^0(r)$ is the single-particle Hamiltonian for transverse particle motion, $p_\perp = (p_x^2 + p_y^2)^{1/2}$ is the transverse particle momentum, $r = (x^2 + y^2)^{1/2}$ is the radial distance from the beam axis, $\omega_f = \text{const.}$ is the transverse frequency associated with the applied focusing field in the smooth-focusing approximation, and $\phi^0(r)$ is the equilibrium space-charge potential determined self-consistently from Poisson’s equation,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi^0}{\partial r}\right) = -4\pi e_b n_b^0,$$

where $n_b^0(r) = \int d^3 p f^0_b(r, p)$ is the equilibrium number density of beam particles. A perfectly conducting wall is located at radius $r = r_w$.

For present purposes, we consider small-amplitude electrostatic perturbations of the form

$$\delta \phi(x, t) = \hat{\delta \phi}(r) \exp(\text{im} \theta + i k_z z - i \omega t),$$

where $\delta \phi(x, t)$ is the perturbed electrostatic potential, $k_z$ is the axial wavenumber, $m$ is the azimuthal mode number and $\omega$ is the complex oscillation frequency, with $Im \omega > 0$ corresponding to instability (temporal growth). Without presenting algebraic details, using the method of characteristics [2, 3], the linearized Poisson equation can be expressed as

$$(1 + \partial/\partial r) \left(r \partial^2 \hat{\delta \phi}(r) - k_z^2 - m^2 \right) \hat{\delta \phi}(r) = -4\pi e_b \int d^3 p \hat{\delta \tilde{f}_b},$$

where

$$\hat{\delta \tilde{f}_b} = e_b \frac{\partial f^0_b}{\partial H_{\parallel}} \hat{\delta \phi} + e_b \left[ \frac{\partial f^0_b}{\partial H_{\perp}} + k_z v_z \frac{\partial f^0_b}{\partial H_{\perp}} \right]$$

$$\times \int_{-\infty}^{t} dt' \hat{\delta \phi}[r'(t')] \exp[i(k_z z - \omega)(t' - t) + \text{im} \theta'(t')]$$

for perturbations about the choice of the anisotropic thermal equilibrium distribution function in Eq. (1). In the orbit integral in Eq. (5), $Im \omega > 0$ is assumed, and $r'(t') = [x'(t')^2 + y'(t')]^{1/2}$ and $\theta'(t')$ are the transverse orbits in the equilibrium field configuration such that $[x'_b(t'), p'_b(t')]$ passes through the phase-space point $(x_\perp, p_\perp)$ at time $t' = t$ [2, 3].

In Eq. (4), we express the perturbation amplitude as

$$\hat{\delta \phi}(r) = \sum \alpha_n \phi_b(r),$$

where $\{\alpha_n\}$ are constants, and the complete set of vacuum eigenfunctions $\{\phi_b(r)\}$ is defined by $\phi_b(r) = A_n J_m(\lambda_n r/r_w)$. Here, $\lambda_n$ is the $n$th zero of Bessel function $J_m(\lambda_n)$, $0$, and $A_n$ is a normalization constant [2]. This gives the matrix dispersion equation

$$\sum \alpha_n D_{n, n'}(\omega) = 0.$$

The condition for a nontrivial solution to Eq. (6) is

$$det\{D_{n, n'}(\omega)\} = 0.$$

* RESEARCH SUPPORTED BY THE U. S. DEPARTMENT OF ENERGY
which plays the role of a matrix dispersion relation that determines the complex oscillation frequency $\omega$ [2].

In the present analysis, it is convenient to introduce the effective depressed betatron frequency $\omega_{\beta \perp}$. It can be shown [3] for the equilibrium distribution in Eq. (1), that the mean-square beam radius $r^2_b = \langle r^2 \rangle = N_b^{-1} 2 \pi \int_{r_w}^{r_b} drr^4 n^2_b(r)$ is related exactly to the line density $N_b = 2 \pi \int_{r_w}^{r_b} drr^4 n^4_b(r)$, and the transverse beam temperature $T_{\perp b}$, by the equilibrium radial force balance equation

$$\omega^2 r^2_b = \frac{N_b \epsilon_b^2}{m_b} + \frac{2 T_{\perp b}}{m_b}. \tag{8}$$

Equation (8) can be rewritten as

$$\left( \omega_f^2 - \frac{1}{2} \omega_{pb}^2 \right) r^2_b = \frac{2 T_{\perp b}}{m_b}, \tag{9}$$

where we have introduced the effective average beam plasma frequency $\tilde{\omega}_{pb}$ defined by

$$r^2_b \omega_{pb}^2 = \int_{r_w}^{r_b} drr^2 \omega_{pb}(r) = \frac{2 \epsilon_b^2 N_b}{m_b}, \tag{10}$$

where $\omega_{pb}(r) = 4 \pi n_b^0(r) c^2 / \gamma_b m_b$ is the relativistic plasma frequency-squared. Then, Eq. (9) can be used to introduce the effective depressed betatron frequency $\tilde{\omega}_{\beta \perp}$ defined by

$$\omega^2_{\beta \perp} = \left( \omega_f^2 - \frac{1}{2} \omega_{pb}^2 \right) = \frac{2 T_{\perp b}}{m_b \varphi^2_b}, \tag{11}$$

and the normalized tune depression $\tilde{\nu} / \nu_0$ defined by

$$\tilde{\nu} \equiv \frac{\omega_{\beta \perp}}{\omega_f} = (1 - \bar{s}_b)^{1/2}. \tag{12}$$

If, for example, the beam density were uniform over the beam cross section, then Eq. (11) corresponds to the usual definition of the depressed betatron frequency for a Kapchinskij – Vladimirskij (KV) [3] beam, and it is readily shown that the radial orbit $\tilde{r}(\tau)$ occurring in Eq. (5) can be expressed as [2]

$$\tilde{r}^2(\tau) = \frac{H_\perp}{m_b \omega_{\beta \perp}^2} \left[ 1 - \sqrt{1 - \left( \frac{\omega_{\beta \perp} P_\perp}{H_\perp} \right)^2 \cos(2 \omega_{\beta \perp} \tau)} \right]. \tag{13}$$

In general, for the choice of equilibrium distribution function in Eq. (1), there will be a spread in transverse depressed betatron frequencies $\omega_{\beta \perp}(H_\perp, P_\perp)$, and the particle trajectories will not be described by the simple trigonometric function in Eq. (13). For present purposes, however, we consider a simple model in which the radial orbit $\tilde{r}(\tau)$ occurring in Eq. (5) is approximated by Eq. (13) with the constant frequency $\omega_{\beta \perp}$ defined in Eq. (11), and the approximate equilibrium density profile is defined by $n_b^2(\tau) = \bar{n}_b \exp(-m_b \omega_{\beta \perp}^2 r^\tau / 2 T_{\perp b})$. For a nonuniform beam, $\omega_{\beta \perp}^{-1}$ is the characteristic time for a particle with thermal speed $v_{th,\perp} = (2 T_{\perp b}/m_b)^{1/2}$ to cross the rms radius $r_b$ of the beam. In this case $D_{n,n'}(\omega)$ can be evaluated in closed analytical form [2] provided the conducting wall is sufficiently far removed from the beam ($r_w/r_b \geq 3$, say). In this case, the matrix elements decrease exponentially away from the diagonal, with

$$\left| \frac{D_{n,n+k}}{D_{n,n}} \right| \sim \exp \left( -\frac{\pi^2 k^2 \beta_{\perp}^2}{4 r_w^2} \right), \tag{14}$$

where $k$ is an integer. Therefore, for $r_w/r_b \geq 3$, we can approximate $\{D_{n,n'}(\omega)\}$ by a tri-diagonal matrix. In this case, for the lowest-order radial modes ($n = 1$ and $n = 2$), the dispersion relation (7) can be approximated by [2]

$$D_{1,1}(\omega) D_{2,2}(\omega) - [D_{1,2}(\omega)]^2 = 0, \tag{15}$$

where use has been made of $D_{1,2}(\omega) = D_{2,1}(\omega)$.

Typical numerical results [2] obtained from the approximate dispersion relation utilizing Eq. (15) are presented in Figs. 1 – 2 for the case where $r_w = 3 r_b$. Only the leading-order nonresonant terms and one resonant term at frequencies $\omega \approx \pm 2 \omega_{\beta \perp}$ for even values of $m$, and $\omega \approx \pm \omega_{\beta \perp}$ for odd values of $m$, have been retained in the analysis [2]. Note from Fig. 1 that the critical values of $k_{\perp} r_w$ for the onset of instability and for maximum growth rate increase as the azimuthal mode number $m$ is increased. As expected, finite $-T_{\perp b}$ effects introduce a finite bandwidth in $k_{\perp} r_w$ for instability, since the modes with large values of $k_{\perp} r_w$ are stabilized by Landau damping [2, 3]. Also, the unstable modes with odd azimuthal number are purely growing.

Note from Fig. 2 that the $m = 1$ dipole mode has the

![Figure 1: Plots of normalized growth rate \((Im \omega)/\omega_f\) and real frequency \((Re \omega)/\omega_f\) versus \(k_{\perp} r_w\) for \(\tilde{\nu}/\nu_0 = 0.53\) and \(T_{\perp b}/T_{\perp b} = 0.02\) [2].](image)

![Figure 2: Plots of normalized growth rate \((Im \omega)_{max}/\omega_f\) and real frequency \((Re \omega)_{max}/\omega_f\) at maximum growth versus tune depression \(\tilde{\nu}/\nu_0\) for \(T_{\perp b}/T_{\perp b} = 0.02\) [2].](image)
rate, decrease with azimuthal mode number \( m \). The instability is absent for \( \nu/\nu_0 > 0.77 \) for the choice of parameters in Fig. 2. The real frequency \( (Re \omega)/\omega_f \) of the unstable modes with odd azimuthal numbers \( m = 1, 3, \cdots \) are zero and are not plotted in Fig. 2. Moreover, the real frequency is plotted only for the unstable modes.

**BEST SIMULATION RESULTS**

Typical numerical results obtained with the linearized version of the 3D BEST code [4] are presented in Figs. 3-5 [2] for the case where \( r_w = 3r_b \) and \( T_{|b|}/T_{\perp b} = 0.02 \), and for perturbations with a spatial dependence proportional to \( \exp(ik_z z + im\theta) \), where \( k_z \) is the axial wavenumber, and \( m \) is the azimuthal mode number. Random initial perturbations are introduced to the particle weights, and the beam is propagated from \( t = 0 \) to \( t = 200\omega_f^{-1} \). Note from Fig. 3 that the instability has a finite bandwidth with maximum growth rate occurring at \( k_z r_w \approx 9 \). From Fig. 4, the critical value of \( \nu/\nu_0 \) for the onset of the instability decreases with azimuthal mode number \( m \). The real frequency \( (Re \omega)/\omega_f \) of the unstable modes for odd azimuthal numbers \( m = 1, 3 \) are zero and are not plotted. Moreover, the real frequency is plotted only for the unstable modes. Consistent with the analytical predictions, note that the dipole mode \( (m = 1) \) has the largest growth rate. Furthermore, all modes are found to be stable in the region \( \nu/\nu_0 \geq 0.85 \). The simulation results presented in Figs. 3 and 4 are in good qualitative agreement with the theoretical model (see Figs. 1 and 2). Moreover, Fig. 5 shows that instability is absent for \( T_{|b|}/T_{\perp b} > 0.08 \).

**Figure 3:** Plots of normalized real frequency \( (Re \omega)/\omega_f \) and growth rate \( (Im \omega)/\omega_f \) versus \( k_z r_w \) for \( \nu/\nu_0 = 0.53 \) and \( T_{|b|}/T_{\perp b} = 0.02 \) [2].

**Figure 4:** Plots of normalized real frequency \( (Re \omega)_{max}/\omega_f \) and growth rate \( (Im \omega)_{max}/\omega_f \) at maximum growth versus normalized tune depression \( \nu/\nu_0 \) for \( T_{|b|}/T_{\perp b} = 0.02 \) [2].

**Figure 5:** Longitudinal threshold temperature \( T_{|b|}^{th} \) normalized to the transverse temperature \( T_{\perp b} \) for onset of instability plotted versus normalized tune depression \( \nu/\nu_0 \) [2].

**CONCLUSIONS**

To summarize, the BEST code [4] was used to investigate the detailed stability properties of intense charged particle beams with large temperature anisotropy \( (T_{|b|}/T_{\perp b} \ll 1) \) for three-dimensional perturbations with several values of azimuthal wave number \( m = 0, 1, 2, 3 \). An analytical model, which generalizes the classical Harris-like instability to the case of an intense charged particle beam with anisotropic temperature, has been developed [2]. Both the simulations and the analytical results clearly show that moderately intense beams with \( s_b \geq 0.5 \) are linearly unstable to short wavelength perturbations with \( k_z^2 r_b^2 \geq 1 \), provided the ratio of longitudinal and transverse temperatures is smaller than some threshold value.

**REFERENCES**


