INTENSE SHEET BEAM STABILITY PROPERTIES FOR UNIFORM PHASE-SPACE DENSITY

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Abstract

A self-consistent one-dimensional waterbag equilibrium \( f_b(x, p_x) \) for a sheet beam propagating through a smooth focusing field is shown to be exactly solvable for the beam density \( n_b(x) \) and space-charge potential \( \phi^0(x) \). A closed Schrodinger-like eigenvalue equation is derived for small-amplitude perturbations, and the WKB approximation is employed to determine the eigenfrequency spectrum as a function of the normalized beam intensity \( s_b = \hat{\omega}^2_{pb}/\gamma^2_b \omega^2_{b\perp} \), where \( \hat{\omega}^2_{pb} = 4\pi n_b e^2_b/\gamma m_b \) is the relativistic plasma frequency-squared and \( \hat{n}_b = n_b(x = 0) \) is the on-axis number density of beam particles.

SHEET BEAM EQUILIBRIUM WITH UNIFORM PHASE-SPACE DENSITY

We consider an intense sheet beam [1], made up of particles with charge \( e_b \) and mass \( m_b \) which propagates in the z-direction with directed kinetic energy \( (\gamma - 1)m_b c^2 \) and average axial velocity \( V_b = \beta_b c = \text{const.} \). Here, \( \gamma_b = (1 - \beta_b^2)^{-1/2} \) is the relativistic mass factor, \( c \) is the speed of light in vacuo, and the beam is assumed to be uniform in the y- and z- directions with \( \partial/\partial y = \partial/\partial z = 0 \). The beam is centered in the x-direction at \( x = 0 \), and transverse confinement is provided by an applied focusing force, \( F_{foc} = -\gamma_b m_b \omega^2_{b\perp} x \), with \( \omega^2_{b\perp} = \text{const} \) in the smooth focusing approximation. The transverse dimension of the sheet beam is denoted by \( 2x_b \), and planar, perfectly conducting walls are located at \( x = \pm x_b \). The particle motion in the beam frame is assumed to be nonrelativistic, and we introduce the effective potential \( \psi(x, t) \) defined by

\[ \psi(x, t) = \frac{1}{2} \gamma_b m_b \omega^2_{b\perp} x^2 + \frac{1}{\gamma_b} e_b \phi(x, t). \]

The Vlasov-Maxwell equations describing the self-consistent nonlinear evolution of \( f_b(x, p_x, t) \) and \( \psi(x, t) \) can be expressed as [2]

\[ \left( \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial p_x} \right) f_b = 0, \]

and

\[ \frac{\partial^2 \psi}{\partial x^2} = \gamma_b m_b \omega^2_{b\perp} - \frac{4\pi e_b^2}{\gamma_b} \int_0^\infty dp_x f_b. \]

As an equilibrium example (\( \partial/\partial t = 0 \)) that is analytically tractable, we consider the choice of distribution function

\[ F_b(H_\perp) = \frac{\hat{n}_b}{8\gamma_b m_b \hat{H}_\perp} \Theta(H_\perp - \hat{H}_\perp), \]

where \( H_\perp = p_x^2/2\gamma_b m_b + \psi^0(x) \) is the transverse Hamiltonian, \( \Theta(x) \) is the Heaviside step-function, and \( \hat{n}_b, \hat{H}_\perp \) are positive constants. Evaluating the number density \( n_b(x) = \int_{-\infty}^\infty dp_x F_b(H_\perp) \), we readily obtain

\[ n_b(x) = \begin{cases} \hat{n}_b \left[ 1 - \psi^0(x)/\hat{H}_\perp \right]^{1/2}, & -x_b < x < x_b, \\ 0, & |x| > x_b. \end{cases} \]

Here, the location of the beam edge \( (x = \pm x_b) \) is determined from

\[ \psi^0(x = \pm x_b) = \hat{H}_\perp, \]

where \( \psi^0(x = 0) = 0 \) is assumed. It is useful to introduce the effective Debye length \( \lambda_D \) defined by

\[ \lambda_D^2 = \frac{\gamma_b^2 \hat{H}_\perp}{4\pi \hat{n}_b e_b^2} = \frac{1}{2} \frac{\gamma_b^2 \hat{\omega}^2_{pb}}{\omega^2_{b\perp}}. \]

Here, \( \hat{\nu}_0 = (2\hat{H}_\perp/\gamma_b m_b)^{1/2} \) is the maximum speed of a particle with energy \( \hat{H}_\perp \) as it passes through \( x = 0 \). Substituting Eq. (5) into Eq. (3) then gives

\[ \frac{\partial^2}{\partial x^2} \left( \frac{\psi^0(x)}{\hat{H}_\perp} \right) = \frac{1}{\lambda_D^2} \left( \frac{1}{s_b} - \left[ 1 - \psi^0(x)/\hat{H}_\perp \right]^{1/2} \right), \]

in the beam interior \( (-x_b < x < x_b) \). Equation (8) is to be integrated subject to the boundary conditions \( \psi^0(x = 0) = 0 = \left[ \partial \psi^0/\partial x \right]_{x=0} \). For physically acceptable solutions to Eq. (8), the condition \( \left[ \partial^2 \psi^0/\partial x^2 \right]_{x=0} > 0 \) imposes the requirement that \( s_b \) lies in the interval \( 0 < s_b < 1 \), where \( s_b = \hat{\omega}^2_{pb}/\gamma_b^2 \omega^2_{b\perp} \). The regime \( s_b \ll 1 \) corresponds to a low-intensity, emittance-dominated beam, whereas the regime \( s_b \rightarrow 1 \) corresponds to a low-emittance, space-charge-dominated beam. In solving Eq. (8), it is convenient to introduce the dimensionless variables defined by

\[ X = \frac{x}{\lambda_D}, \quad \tilde{\psi}^0(X) = \frac{\psi^0(x)}{\hat{H}_\perp}. \]

Substituting Eq. (9) into Eq. (8), integrating once, and enforcing \( \left[ \psi^0 \right]_{x=0} = 0 = \left[ \partial \psi^0/\partial x \right]_{x=0} \), gives

\[ 1 \frac{\left( \frac{d \tilde{\psi}^0}{d X} \right)^2}{2} = \frac{1}{s_b} \tilde{\psi}^0 + \frac{2}{3} \left( 1 - \tilde{\psi}^0 \right)^{3/2} - 1 \]

in the interval \( -x_b/\lambda_D \leq X \leq x_b/\lambda_D \). Equation (10) can be integrated exactly to determine \( X \) as a function of \( (1 - \tilde{\psi}^0)^{1/2} = n_b^0(X)/\hat{n}_b \) [see Eq. (5)]. We express \( X = \).
\[ \int_0^\infty d\tilde{\omega}^0/(d\tilde{\omega}^0/dX) \text{, change variables to } z = (1 - \tilde{\omega}^0)^{1/2} \text{, and make use of Eq. (10). This gives [1, 3]} \]
\[ X = 3^{1/2} \int_{(1 - \tilde{\omega})^{1/2}}^1 z dz \left[ (1 - z)(a^+ - z)(z - a^-) \right]^{1/2}, \]  
(11)

where \( a^+ \) and \( a^- \) are defined by
\[ a^\pm = \frac{1}{4s_b} \left( 3 - 2s_b \pm \sqrt{3 + 4s_b - 4s_b^2} \right)^{1/2}. \]  
(12)

From Eqs. (6) and (11) we obtain a closed expression for \( x_b/\lambda_D \) in terms of the normalized beam intensity \( s_b \) for the choice of equilibrium distribution function in Eq. (4). The areal density of the beam particles, \( N_b = \int_{-x_b}^{x_b} dx n_b^0(x) \), for the density profile in Eq. (5) can be expressed as
\[ N_b = 2\hat{n}_b \int_0^{x_b} dx [1 - \psi^0(x)/\hat{H}_\perp]^{1/2}. \]  
(13)

Some algebraical manipulation that make use of Eqs. (9), (10) and (13) gives
\[ \frac{N_b}{2\hat{n}_b x_b} = 3^{1/2} \lambda_D x_b \int_0^1 \frac{z^2 dz}{[1 - (z)(a^+ - z)(z - a^-)]^{1/2}}, \]  
(14)

where \( x_b/\lambda_D \) is determined from Eq. (11). Note that \( N_b/2\hat{n}_b x_b \) depends only on the dimensionless intensity parameter \( s_b \). Typical normalized density profiles

\[ 2x_b n_b^0(x)/N_b \text{ versus } x/x_b \text{ for different values of the normalized beam intensity } s_b \text{ corresponding to (a) } s_b = 0.2, \]
\[ (b) s_b = 0.9, (c) s_b = 0.99, (d) s_b = 0.999, (e) s_b = 0.999999. \]

\[ 2x_b n_b^0(x)/N_b \] are illustrated in Fig.1 for values of \( s_b \) ranging from \( s_b = 0.2 \) to \( s_b = 0.999999 \) [1]. Finally, defining the equilibrium transverse pressure profile by \( P_b^0(x) = \int_{-\infty}^\infty dp_x (p_x^2/\gamma_b m_b)^0 \), we readily obtain
\[ P_b^0(x) = \frac{4}{3} \hat{n}_b \hat{H}_\perp \left[ 1 - \frac{\psi^0(x)}{\hat{H}_\perp} \right]^{3/2}. \]  
(15)

Comparing Eqs. (5) and (15), note that \( P_b^0(x) = const [n_b^0(x)]^3 \), which corresponds to a triple-adiabatic pressure relation.

**LINEARIZED EQUATIONS AND STABILITY ANALYSIS**

The *linearized* Vlasov-Maxwell equations can be expressed as [2]
\[ \left( \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} - \frac{\partial \psi^0}{\partial x} \frac{\partial}{\partial p_x} \right) \delta f_b = v_x \frac{\delta \psi}{\partial x} \frac{\partial F_b}{\partial \psi}, \]  
(16)

and
\[ \frac{\partial^2}{\partial x^2} \delta \psi = -\frac{4\pi e^2}{m_b^0} \delta n_b, \]  
(17)

where \( \delta n_b(x, t) = \int_{-\infty}^\infty dp_x \delta f_b \) is the perturbed number density of beam particles. In analyzing Eqs. (16) and (17), it is convenient to change variables from \((x, p_x, t)\) to the new variables \((x', H_\perp, \tau)\) defined by
\[ x' = x, \quad \tau = t, \quad H_\perp = \frac{1}{2\gamma_b m_b} p_x^2 + \psi^0(x). \]  
(18)

Substituting Eqs. (18) into Eqs. (16) and (17) gives for the evolution of the perturbations \( \delta f_b(x', H_\perp, \tau) \) and \( \delta \psi(x', \tau) \),
\[ \left( \frac{\partial}{\partial \tau} + v_x \frac{\partial}{\partial x'} - \frac{\partial \psi^0}{\partial x'} \frac{\partial}{\partial H_\perp} \right) \delta f_b = v_x \frac{\delta \psi}{\partial x'} \frac{\partial F_b}{\partial \psi}, \]  
(19)
\[ \frac{\partial^2}{\partial x'^2} \delta \psi = -\frac{4\pi e^2}{\gamma_b^2} \delta n_b. \]  
(20)

In Eq. (19), \( v_x = +v(H_\perp, x') \) for the forward-moving particles with \( v_x > 0 \), and \( v_x = -v(H_\perp, x') \) for the backward-moving particles with \( v_x < 0 \), where
\[ v_x = \pm v(H_\perp, x') \equiv \pm \left( \frac{2H_\perp}{\gamma_b m_b} \right)^{1/2} \left[ 1 - \psi^0(x') \right]^{1/2}. \]  
(21)

Furthermore,
\[ \frac{\partial F_b}{\partial H_\perp} = -\frac{\hat{n}_b}{2\gamma_b m_b v_0} \delta (H_\perp - \hat{H}_\perp), \]  
(22)

where \( \hat{v}_0 = (2\hat{H}_\perp/\gamma_b m_b)^{1/2} \). Using Eqs. (19)-(22) and introducing \( \delta E_x(x', \tau) = -(\partial/\partial x') \delta \phi(x', \tau) = -(\gamma_b^2/e_b)(\partial/\partial x') \delta \psi(x', \tau) \), after some algebra manipulation we obtain [1]
\[ \frac{\partial^2}{\partial x'^2} \delta E_x - \frac{\hat{v}_0^2}{\gamma_b^2} N'(x') \frac{\partial}{\partial x'} \left[ N(x') \frac{\partial}{\partial x'} \delta E_x \right] \]  
\[ = -\frac{\hat{v}_0^2}{\gamma_b^2} N(x') \delta E_x, \]  
(23)

where \( N(x') \) is the (dimensionless) profile shape function defined by
\[ N(x') = \left[ 1 - \psi^0(x') \right]^{1/2}. \]  
(24)

In the analysis of Eq. (23), we make use of a normal-mode approach and express \( \delta E_x(x', \tau) = \)
\[ \delta E_x(x', \omega) \exp(-i\omega \tau), \] where \( \omega \) is the (generally complex) oscillation frequency. Equation (23) can be represented in a convenient form by introducing the angle variable \( \alpha \) defined by

\[ \alpha = \frac{\pi}{2} \frac{X'}{X_b} = \frac{\omega_0}{\nu_0} x', \tag{25} \]

where \( X' \) and \( \omega_0 \) are defined by

\[ X' = \int_0^{x'} \frac{dx'}{N(x')}, \quad \omega_0 = \frac{\pi}{2} \frac{\nu_0}{X_b}, \tag{26} \]

where \( X_b = X'(x_b) \). Substituting Eq. (25) into Eq. (23) gives the eigenvalue equation

\[ \frac{\partial^2}{\partial \alpha^2} \delta E_x + \left[ \omega^2 - \frac{\omega_b^2}{X_b} \right] \delta E_x = 0. \tag{27} \]

Equation (27) is to be solved over the interval \([-\pi/2 < \alpha < \pi/2]\) subject to the boundary conditions \( \delta E_x(\alpha = \pm \pi/2, \omega) = 0 \). Substituting Eqs. (10) and (24) into Eq. (25) gives

\[ \alpha = \frac{\pi}{2} \frac{\lambda_D}{X_b} \frac{3^{1/2}}{L \int_N} \frac{dz}{[(1 - z)(a^+ - z) - a^-]^{1/2}}, \tag{28} \]

where \( a^\pm \) is defined in Eq. (12). Some algebraic manipulation gives exactly for the inverse function \( N(\alpha) \)

\[ N(\alpha) = \left[ 1 - a^+ a^- \frac{\alpha X_b}{\pi \lambda_D} \frac{[a^+ - a^-]}{3} \right]^{1/2}, \tag{29} \]

where \( sn(\beta, \kappa) \) is the Jacobi elliptic sine function and \( \kappa = [(1 - a^+)/a^+ - a^-]^{1/2} \). In Eqs. (28)-(29), the "stretched" half-layer thickness \( X_b \) measured in units of the Debye length \( \lambda_D \) is given by

\[ X_b = \frac{2 \cdot 3^{1/2}}{(a^+ - a^-)^{1/2}} F \left( \arcsin \left( \frac{\kappa^2}{a^+} \right)^{-1/2}, \kappa \right), \tag{30} \]

where \( F \) is the elliptic integral of the first kind. Using the expression for \( N(\alpha) \) in Eq. (29), the eigenvalue equation (27) can be solved numerically for \( \delta E_x(\alpha, \omega) \) and the eigenvalues \( \omega^2 \) subject to the boundary conditions \( \delta E_x(\alpha = \pm \pi/2, \omega) = 0 \). An approximate expression for the eigenvalues of the Schrödinger-like equation (27) can be obtained in the WKB approximation. The Born-Zommerfeld formula, when applied to Eq. (27), gives

\[ \frac{\omega_b}{\gamma_b \omega_0} \int_{-\pi/2}^{\pi/2} d\alpha \left[ \left( \frac{\gamma_b \omega_m}{\omega_b} \right)^2 - N(\alpha) \right]^{1/2} = \pi m, \tag{31} \]

where \( \omega_m \) is the \( m \)-mode eigenfrequency with \( m \) half-wavelength oscillations of \( \delta E_x \) over the layer thickness.

Making use of Eq. (28), the result in Eq. (31) can be rewritten as

\[ 6^{1/2} \int_0^1 \frac{dz (q_m^2 z - 1)^{1/2}}{(1 - z)(a^+ - z)(z - a^-)^{1/2}} = \pi m, \tag{32} \]

where \( q_m \) and \( r \) are defined by \( q_m = \omega_m / (\hat{\omega}_b/\gamma_b) \) and \( r = \kappa [(q_m^2 - a^+)/(q_m^2 - 1)]^{1/2} \). Equation (32) has been solved numerically \([1]\) for \( \omega_m^2 \), and the results have been compared with the numerical solutions of the eigenvalue equation (27) (Fig. 2). In Fig. 2, the convention is such that there are \( m \) half-wavelength oscillations of \( \delta E_x \) over the layer thickness. Note that low beam intensity \( (s_b \ll 1) \) corresponds to \( \nu / \nu_0 \rightarrow 1 \), with \( \omega_m \approx m \omega_{\beta \perp} \), whereas the space-charge-dominated regime \( (s_b \rightarrow 1) \) corresponds to \( \nu / \nu_0 \rightarrow 0 \), with \( \omega_m \approx \omega_{\beta \perp} \approx \hat{\omega}_b / \gamma_b \).

To summarize, we have demonstrated that the self-consistent waterbag equilibrium \( f^0_b \) satisfying the steady-state \((\partial / \partial t = 0) \) Vlasov-Maxwell equations is exactly solvable for the beam density \( f_b^0(x) \) and electrostatic potential \( \phi^0(x) \). In addition, we derived a closed Schrödinger-like eigenvalue equation for small-amplitude perturbations \( (\delta f_b, \delta \phi) \) about the self-consistent waterbag equilibrium in Eq. (4). In the eigenvalue equation, the density profile \( n_b^0(x) \) plays the role of the potential \( V(x) \) in the Schrödinger equation. The eigenvalue equation was investigated analytically and numerically, and the eigenfrequencies were shown to be purely real.

**REFERENCES**


[3] The integrals in Eqs. (11), (14), (28), (32) can be expressed in terms of elliptic functions (see Ref. 1).