1. Introduction

Coherent dipole longitudinal instabilities have been observed in the Electron-Positron Accumulation Ring (EPA), which is part of the LEP injector chain. These instabilities turn into a strong limitation upon the maximum beam current, depending on RF cavity voltage and tuning angle. Although the nominal performances of the machine are not affected, as far as the operation of the LEP injector chain is concerned, a study of beam loading effects seems worthwhile, because: i) it might be necessary to accumulate up to 4 times the nominal intensity for LEP injection, ii) it was foreseen to reduce the cavity voltage for optimum injection to the PS, iii) owing to the rather unconventional design of the RF cavity, one might expect the intensity limits to be different from those predicted by classical Robinson's criterion.

In this paper a transformer-coupled resonator model for the EPA RF cavity is presented, and a detailed analysis of its beam loading stability is performed. The results are compared both with Robinson's criterion and with some measurements taken during EPA running-in.

2. The cavity model

The EPA RF system [1] consists of an accelerating cavity coupled through a magnetic loop to an amplifier cavity where the power tetrode is located. The equivalent lumped circuit of this system is shown in Fig. 1, together with the usual phasor diagram. The power tube is represented by a current generator with its plate resistance \( R \), added in parallel. The beam is also represented by a current generator at the fundamental frequency 19.1 MHz, whose amplitude is twice the DC beam current (valid for typical EPA bunch lengths). In the following we shall adopt the notation used by Pedersen [2].

Using Kirchhoff's laws the following complex quantities can be calculated:

\[ Z_{11} = \left( \frac{V_1}{I_1} \right)_{h=0}, \quad \text{i.e. the impedance seen by the power generator}, \]

\[ Z_{21} = \left( \frac{V_2}{I_1} \right)_{h=0}, \quad \text{i.e. the anode impedance transformed to the accelerating gap}, \]

\[ Z_{2z} = \left( \frac{V_2}{I_2} \right)_{h=0}, \quad \text{the gap impedance (as seen by the beam)} \]

where \( s = jo \). The damping coefficients \( \alpha_{1,2} = \frac{1}{R_{1,2} C_{1,2}} \), the mutual inductance coefficient \( k = \frac{M}{\sqrt{L_{1,2} L_{2,1}}} \) and the resonant frequencies \( \omega_{1,2} = \frac{1}{\sqrt{L_{1,2} C_{1,2}}} \) of the primary and secondary circuits are derived from low-level measurements.

A typical set of parameters is reported in Table I. The plate resistance has been estimated from the tube characteristics (SIEMENS RS 1084 tetrode) and added in parallel on the generator side. The cavity shunt impedance \( R_c \) has not been directly measured, but it is estimated as \( R_c = Z_n Q \), where the characteristic impedance \( Z_n = 41 \Omega \) was calculated by SUPERFISH and the loaded quality factor \( Q = 3446 \) was measured with a Network Analyzer.

Table I - Cavity parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1 = \frac{\omega_1}{2\pi} )</td>
<td>16857.0 kHz, ( Q_1 = 1000 ), ( Z_n = 45 \Omega )</td>
</tr>
<tr>
<td>( f_2 = \frac{\omega_2}{2\pi} )</td>
<td>19029.34 kHz, ( Q_2 = 6000 ), ( Z_n = 41 \Omega )</td>
</tr>
<tr>
<td>( R_p = 7.2 , k\Omega )</td>
<td>( f_0 = \frac{\omega_0}{2\pi} = 19085.24 , kHz ) bunch frequency</td>
</tr>
<tr>
<td>( \frac{\Omega}{2\pi} = \frac{\omega_k}{4\pi} )</td>
<td>335.5 kHz</td>
</tr>
</tbody>
</table>

The stability analysis of such a system will follow the guidelines of Pedersen's work. For the beam we assume rigid bunches, so its phase transfer function between excitation and beam is:

\[ B(s) = \frac{\omega}{s^2 + \omega_1^2} \]

The impedance (1) has been used to calculate the transfer functions \( C_{BP}(s) \) and \( C_{PB}(s) \) for transmission of amplitude and phase.
modulations through the cavity, which are defined in Appendix A.

The characteristic equation is:

\[ 1 - B(s) \cdot \left( G_{\text{out}}^{B}(s) + \tan \phi_{s} G_{\text{in}}^{B}(s) \right) = 0 \]  

and we know that the system is unstable if, and only if, the characteristic equation has roots with positive real part.

By inspection of the above formulae, it is easily recognized that this equation will be of 10th degree in \( s \), with very complicated coefficients, making the analytical solution impossible even with the help of symbolic programming. Therefore a program has been written to perform this calculation numerically (see Appendix B).

3. Results and comparison with measurements

First we have considered the case where the generator current \( I_{G} \) is constant. If the power tube is assumed to be an ideal current generator, this corresponds to the experiment where a constant excitation is applied to the control grid of the tube. Furthermore the tuning loop is disabled in order to keep the phase angle \( \phi_{L} \) of the cavity impedance constant. The parameters which enter eq. (2), are \( I_{G}, I_{B}, \phi_{Z}, \phi_{R} \) and \( \phi_{L} \). These two last angles are no longer constant during accumulation, so we used the steady state conditions as derived from the phasor diagram:

\[
\begin{align*}
V & = I_{C} \cos \phi_{L} = \frac{I_{B} R_{s}}{V_{c}} \cos \phi, \\
I_{C} \sin \phi_{L} + I_{B} \sin \phi_{s} \tan \phi_{L} & = \frac{I_{B} R_{s}}{V_{c}} \sin \phi
\end{align*}
\]

with \( \phi_{s} = \arccos \left(-\sqrt{1 - \left(\frac{U_{0}}{eV_{c}}\right)^{2}}\right) \) and \( U_{0} \) is the synchrotron radiation loss per turn at 500 MeV. From these 3 eqs. we eliminate \( \phi_{s} \) and \( \phi_{L} \) and we get a 4th order polynomial (see Appendix C) in \( V_{c} \) which is analytically solvable. In this way we determine the stability for any \( \phi_{s} = \text{const.} \) trajectory in the \( (\phi_{L}, I_{B}) \) plot by applying the Routh-Hurwitz criterion to the characteristic equation.

In Fig. 3 the instability zones for the single resonator model are displayed, as calculated by the program, together with the stable trajectories of Fig. 2. The areas shaded with ++ are unstable according to the Routh-Hurwitz criterion, while those shaded with squares are forbidden by power limits.

Looking at Fig. 3 the only difference between the two models would appear if the instability limit were occurring before the end of the trajectory. This is not the case for the parameters of the cavity.

There is a remarkable good agreement between the computed and the experimental curves if we plot, instead of \( \phi_{L} \), the cavity voltage \( V_{c} \) against the beam current as in Fig. 4. This is a confirmation of the validity of our model, since the voltage measurement was cross-checked by various means.

We have also investigated the stability of the transformer-coupled system when the cavity voltage is given. In Fig. 5 the trajectories at \( V_{c} = 10 \) kV and constant tuning angle are displayed together with the instability domains. The Robinson limits are superimposed and show good agreement.
Fig. 5 - Stability plot for the case $V_c = 10$ kV: + Robinson instability, -- dashed line: theoretical limit, - solid lines: trajectories.

**Conclusions**

The two-resonator model was introduced as a possible explanation of the observed beam intensity limits in EPA. As shown in Fig. 4 these limits are very close to the theoretical predictions. These predictions appear to be almost identical to those of the single resonator model as shown in Fig. 3, at least for the EPA cavity parameters, where the two resonance peaks are about 2 MHz apart.

**Acknowledgements**

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**References**


**APPENDIX A - Transfer functions through cavity**

$Z$ being given by eq.(1)

$$G_s(s) = \frac{1}{2} \left[ \frac{Z(s + j\omega)}{Z(j\omega)} - \frac{Z(s - j\omega)}{-Z(-j\omega)} \right]$$

$$G_c(s) = \frac{1}{2} \left[ \frac{Z(s + j\omega)}{Z(j\omega)} - \frac{Z(s - j\omega)}{-Z(-j\omega)} \right]$$

with $U = I_G/I_B$ and $\phi = \phi_s - \phi_L + \frac{\pi}{2}$

$$G_{pp}^R = \frac{G_s(s)(1 + U \cos\phi) - G_c(s) U \sin\phi)}{[1 + U^2 + 2U \cos\phi]}$$

$$G_{pp}^R = \frac{G_s(s) U \sin\phi + G_c(s)(1 + U \cos\phi))}{[1 + U^2 + 2U \cos\phi]}$$

**APPENDIX B - Calculations of the coefficients of the characteristic equation**

The characteristic equation results as a linear combination of terms $s^n N(s + s_k) \cdot D(s + s_i)$, where $s_k$ and $s_i$ are $j\omega_c$ or $-j\omega_c$, $n$ is 0 or 2, and N and D are the 3rd and 4th order polynomials found in $Z_{32} = N/D$ (eq. 1). It is not difficult to calculate the coefficients of $s^m$, by summing all $n_k e^{k \phi} e^m$ such that $k + l + n = m$, and, further, to add the coefficients coming from all the N D products.

**APPENDIX C - Polynomial in $V_c$**

$$\left[ \frac{V_c}{R_s \cos\phi} \right]^2 + 2 \left( \frac{V_c}{R_s \cos\phi} \right) \left[ 2 U B \frac{U_B}{R_s} + 2 B_2 - 2 B_1^2 \sin^2\phi \right] +$$

$$\left[ 2 U_B \frac{U_B}{R_s} + (B_2 - B_1^2) \right]^2 + \left[ 2 U B_1 \tan\phi \right]^2 = 0$$