CANCELLATION EFFECTS IN CSR INDUCED BUNCH TRANSVERSE DYNAMICS IN BENDS

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1 INTRODUCTION

The existence of the centrifugal space charge force 
was first pointed out by Talman [1] when studying
the curvature induced transverse force on a coasting beam
in a storage ring. The logarithmic dependence of this force
on particle transverse offset could cause shifts in horizontal
tune and contribute significantly to chromaticity for a
coasting beam. These effects of logarithmic divergence in
on transverse dynamics were later pointed out by
Lee [2] to be cancelled by the effect of beam induced elec-
tric potential, which enters into the transverse dynamics
through dispersion by changing the kinetic energy of the
particles. As the result of the cancellation, the residual ef-
fect on a coasting beam is about \(\sigma_x/R\) times the \(CSCF\)
effect, where \(\sigma_x\) is the transverse bunch size, and \(R\) is the
equilibrium radius of the ring.

Even though the cancellation effect was cleared for
coasting beams, it was a dispute again for the CSR induced
transverse effect for bunched beams. In Ref. [3], it was
concluded that for bunched beams, the effect of \(CSCF\) is
no longer cancelled by the potential energy, and there exists
a longitudinal force named non-inertial space charge force
\(F^{NSCF}\) in addition to the usual longitudinal space charge
and CSR forces. On the other hand, Derbenev pointed out
[4] that for bunched beams, there is always the cancellation
between the effect of \(CSCF\) on the transverse bunch dy-
namics and the effect of potential energy. Further analysis
[5] shows that the accumulated effect of \(F^{NSCF}\) contributes
to the potential energy; thus its effect on the transverse dy-
namics nearly cancels with that of \(CSCF\). Most recently,
the generality of the cancellation effect was questioned [6]
and what exactly the cancellation is meant was under dis-
pute again.

In this paper, we seek to clarify the meaning of the
“cancellation effect” and its general application. By an-
alyzing the generalized momentum and its dynamics, we
show that the “centrifugal space charge force” arises as a
result of the dependence of the metric on coordinates; in
this sense it shares similar geometric feature as that of the
usual centrifugal force. It turns out that for a charged par-
cle in a bunch on a circular orbit, the usual centrifugal
force—which is related to the kinetic momentum—always
works together with the “centrifugal space charge force”,
and jointly they form the generalized centrifugal force—
which is related to the generalized momentum. We show
that in this generalized centrifugal force, the effect of the
“centrifugal space charge force” is always cancelled by the
potential energy effect; as a result, the effective terms after
cancellation is free from logarithmic singularities caused
by the nearby particle interaction. For both steady state and
transient regimes, this cancellation is demonstrated using
numerical simulation, and the behaviors of effective terms
are presented.

2 PARTICLE DYNAMICS IN A ROTATING FRAME

Let us start with the Lagrangian of a particle in an elec-
tron bunch experiencing external and self-interaction fields,
and then derive the dynamical equation of generalized mo-
momentum for particles moving on a circular orbit. In this
way we illustrate how the “centrifugal space charge force”
enters into the picture.

First, for a Cartesian coordinate system with 4-vectors
\(q = (\sigma, r)\), \(U = dq/dr = (\gamma c, \gamma u)\) and 4-potential
\(A = (\Phi, A)\), the covariant form of the Lagrangian with
Minkowski spacetime metric tensor \(g_{\mu\nu}\) is
\[
L = -mc\sqrt{g_{\mu\nu}U^\mu U^\nu + L_{\text{int}}} \tag{1}
\]
with the interaction Lagrangian
\[
L_{\text{int}} = -\frac{e}{c} g_{\mu\nu} A^\nu = -e\gamma (\Phi - \beta \cdot A). \tag{2}
\]
The Euler-Lagrangian equation is
\[
\frac{d}{dr} \frac{\partial L}{\partial \dot{q}^\mu} - \frac{\partial L}{\partial q^\mu} = 0, \tag{3}
\]
where the generalized momentums \(P^\mu = -g^{\mu\nu} \partial L/\partial U^{\nu}\) are
\[
P^0 = (\gamma mc^2 + e\Phi)/c, \quad P = \gamma m u + eA/c. \tag{4}
\]
Let \(dt = \gamma d\tau\); we obtain the 3-dimensional projection of
Eq. (3)
\[
\frac{dP}{dt} = -e(\nabla\Phi - \beta_i \nabla \cdot A_i), \tag{5}
\]
and energy relation from the zeroth component of Eq. (3)
\[
\frac{d(\gamma mc^2 + e\Phi)}{dt} = e \left( \frac{\partial \Phi}{\partial t} - \beta \cdot \frac{\partial A}{\partial t} \right). \tag{6}
\]
The above discussion is based on a Cartesian coordi-
nate system. Next we consider a bunch moving on a cir-
cular orbit. The particle dynamics in the bending plane
is then expressed in terms of the cylindrical coordinates
with respect to the center of the designed circular orbit:
\(r = re_r + r\theta e_\theta, u = re_r + r\theta e_\theta\). The relativistic La-
grangian in terms of cylindrical coordinates is then
\[
L = -mc^2 \sqrt{1 - \frac{r^2 \dot{\theta}^2 + \dot{r}^2}{c^2}} - e(\Phi - \frac{\dot{r}}{c} A_r - \frac{r\dot{\theta}}{c} A_\theta). \tag{7}
\]
To compare with Eq. (5), we rewrite the Euler-Lagrangian equation from Eq. (7) in the following form:

$$\frac{dP_r}{dt} - \nu_s P_s = -e \left( \frac{\partial \Phi}{\partial r} - \beta \cdot \frac{\partial A}{\partial r} \right), \quad (8)$$

$$\frac{dP_s}{dt} + \nu_r P_r = -e \left( \frac{\partial \Phi}{r \partial \theta} - \beta \cdot \frac{\partial A}{r \partial \theta} \right), \quad (9)$$

where the generalized momentums $P_r$ and $P_s$ are defined as

$$P_r = p_r + eA_r/c, \quad P_s = p_s + eA_s/c \quad (10)$$

with the kinetic momentums $p_r = \gamma mv_r$ and $p_s = \gamma mv_\theta$. Comparing to Eq. (5), the right-hand sides in Eqs. (8) and (9) are the projection of the driving term in Eq. (5) to the basis of the rotating frame $(e_r, e_s)$, while the second terms on the left-hand sides in Eqs. (8) and (9) are purely due to the dependence of the metric on coordinates in the rotating coordinate system. Note that Eqs. (8) and (9) are readily reduced to Eq. (5) in straight sections where $r \to \infty$.

For a bunch with design energy $E_0 = \gamma_0 mc^2$ circulating on an orbit with design radius $R$, one has

$$B_{ext} = \frac{mc^2}{e} \frac{\gamma_0^2}{R} (e_s \times e_r), \quad \frac{eA_{ext}}{c} = -\gamma_0 m_0 c R r, \quad (11)$$

where $\beta_0 = (1 - \gamma_0^{-2})^{1/2}$. With $A_s = A_s^{ext} + A_s^{eff}$, the particle’s transverse dynamics can be obtained from Eq. (8)

$$\frac{d\gamma v_r}{dt} = \beta_c c \left( \frac{P_s}{r} - \frac{\gamma_0 mc^2}{R} \right) + F^{eff}. \quad (12)$$

Using $\Phi$ and $A$ to represent only self-interaction potentials from now on, the part of the generalized momentum relating only to bunch self-interaction in Eq. (11) is

$$\tilde{P}_s = \gamma mc^2 \beta_0 + eA_s/c, \quad (13)$$

and the effective radial force $F_r^{eff}$ in Eq. (11) is

$$F_r^{eff} = -e \left( \frac{\partial \Phi}{\partial r} - \beta \cdot \frac{\partial A}{\partial r} \right) - e \frac{dA_r}{cdt}. \quad (14)$$

Here we define $\beta_c e \tilde{P}_s/r$ in Eq. (11) as the “general centrifugal force”

$$F_{GCF} = \beta_c e \tilde{P}_s = \gamma mv_r^2 + F^{CSCF}, \quad (15)$$

where $\gamma mv_r^2$ is the usual centrifugal force, and $F^{CSCF}$ is the “centrifugal space charge force” due to the particles’ collective interaction

$$F^{CSCF} = e \beta_s A_s/r. \quad (16)$$

Even though here $F_r^{tot}$ is dominated by $F^{CSCF}$ and thus is centrifugal in direction, $F^{CSCF}$ is singled out as the “centrifugal space charge force” due to its geometrical nature.

Next we show that in $\tilde{P}_s$ of Eq. (12), the term $eA_s/c$—which represents the effect of $F^{CSCF}$—always works counteractively with the potential energy effect. With the definition of effective parallel force (parallel to $\beta_r e_r + \beta_s e_s$)

$$F^{eff} = e \left( \frac{\partial \Phi}{cdt} - \beta \cdot \frac{\partial A}{cdt} \right), \quad (17)$$

Eq. (6) becomes

$$\frac{dmc^2}{c dt} = \beta \cdot F = -e \frac{d\Phi}{cdt} + F^{eff}, \quad (18)$$

which can be integrated as

$$\gamma mc^2 = \gamma_0 mc^2 + \Delta E^{tot}(t_0) + \int_{t_0}^{t} F^{eff}(t') c dt' - e\Phi(t), \quad (19)$$

with $\Delta E^{tot}(t_0)$ the initial kinetic and potential energy deviation from design energy

$$\Delta E^{tot}(t_0) = [\gamma(t_0) mc^2 - \gamma_0 mc^2] + e\Phi(t_0). \quad (20)$$

As a result, we have by combining Eq. (19) with Eq. (12)

$$\tilde{P}_s = \beta_s \gamma_0 mc + \beta_c \Delta E^{tot}(t_0)/c + \left[ \beta_s \gamma_0 mc - \beta_c \gamma_0 mc \right] c_0(t - \beta_s \Phi), \quad (21)$$

with $\Delta E^{tot}(t_0)$ given in Eq. (20). Applying Eq. (21) to Eq. (11), one gets the equation of motion which contains clearly the $(A_s - \beta_s \Phi)$ term:

$$\frac{d\gamma v_r}{dt} - \beta_c c \left( \frac{\beta_s \gamma_0 mc}{r} - \frac{\beta_c \gamma_0 mc}{R} \right) = G_0 + G_c + G_{||} + G_r \quad (22)$$

with $G_0 = \beta_s^2 \frac{\Delta E^{tot}(t_0)}{r}, \quad G_{||} = \frac{\beta_s^2}{r} \int_{t_0}^{t} F^{eff}(t') c dt', \quad G_c = e \beta_s A_s - \beta_c \Phi, \quad G_r = F^{eff}$

Note that Eq. (22) does not contain any approximation, which shows that the transverse dynamics of an electron is driven by the initial total energy deviation from design energy ($G_0$ term), the effective forces ($G_{||}$ and $G_r$ terms), and the residual of $(A_s - \Phi)$ ($G_c$ term). It should be emphasized that the “cancellation effect” means $A_s$ and $\Phi$ in $G_c$ is nearly cancelled, where the $A_s$ term represents the effect of $F^{CSCF}$. Typically, $A_s$ and $\Phi$ in $G_c$ have logarithmic dependence on the particle’s transverse offset due to local (immediate neighbor) interaction. However, the residual term after their cancellation, $G_c$, is free from the logarithmic singularity. Our simulation in the next section shows that $G_c$ is always negligible compared to $G_{||}$ and $G_r$.

It is interesting to note that for the transient regime of a line bunch entering a circle, $-d\Phi/dt$ does not exhibit logarithmic behavior [7, 8]; neither does $\Phi(t) - \Phi(t_0)$. In this case, apart from the $\gamma^{-2}$ dependence, $\Phi(t_0), \Phi(t)$ and $A_s(t)$ have the same logarithmic behavior; therefore their differences are free from the logarithmic singularity. The
initial potential $\Phi(t_0)$ enters into $G_0$ of Eq. (22), which acts as the initial energy spread, and does not directly cause emittance growth for an achromatic bending system. However, for a non-achromatic bending system such as a single bend in a spectrometer, $\Phi(t_0)$ in $G_0$ can cause some observable effects on particles’ transverse position, and one should be careful with data analysis in such cases. In general, when the bunch is not rigid in bends (such as in Fig. 1), $\Phi(t)$ and $\Phi(t_0)$ no longer have the same logarithmic behavior, while it can be shown analytically that $A_s(t)$ and $\Phi(t)$ are always the same in logarithmic dependence. Therefore the cancellation of logarithmic dependences of $A_s(t)$ and $\Phi(t)$ in $G_c$ always holds.

3 SIMULATION RESULTS

The cancellation effect and behavior of residual terms in a steady state case have been analyzed earlier [5]. Here we use simulation [5] to show how it works in both steady state and transient regimes, including a bunch entering a circle from a straight path or exiting a circle to a straight path. For this purpose, we let a 5 GeV electron bunch with Gaussian longitudinal distribution and rms bunch length 0.2 mm move from a straight path to a 10 m radius. The bunch charge is 1 nC. After $L=2$ m (11.5 deg) of bending, the bunch exits onto a straight path again. The numerical results of various force terms across the bunch at $L=1.6$ m are displayed in Fig. 1, where the bunch transverse size is 72 $\mu$m. The spread of $F^\text{tot}$ and $F^{\text{CSCF}}$ in Fig. 1 for fixed $s/\sigma_s$ is due to their rapid dependence on transverse offset originated from the logarithmic behavior, which was a big concern for coating beams [1] and later was proved to be not effective [2]. It is shown in Fig. 1 that for a bunched beam the effective centripetal radial force $F^\text{eff}_{r}$ has negligible dependence on transverse offset. The $G_c$ term has magnitude of $4 \times 10^{-3}$ keV/m, so it appears to be zero in Fig. 1. In Fig. 2, we track the forces following a single particle in the bunch to show the transient behaviors of the force terms. Here in order to show a clean steady state result (so the particle potential does not change due to its internal motion), we choose a rigid Gaussian line bunch, and the particle dynamics does not respond to CSR force. Here again $G_c$ is practically zero through all the transient and steady state regimes, indicating cancellation of the effect of $F^{\text{CSCF}}$ with the potential energy effect. In Fig. 3, we show that in Eq. (13), $-edA_r/cdt$ is almost discontinuous at entrance and exit of a circle, and it remains zero in steady state, while $\Phi_c$ is negligible. $G_c$ or $F^\text{eff}_r$ may have non-negligible effect on bunch transverse dynamics.

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4 REFERENCES