RADIAL-SECTOR CYCLOTRONS WITH DIFFERENT HILL AND VALLEY FIELD PROFILES

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Abstract
A new class of isochronous cyclotron is described in which more general radial field profiles \( B(r) \) are allowed than the simple proportionality to total energy found in conventional radial- and spiral-sector cyclotrons. Isochronism is maintained by using differently shaped field profiles in the hills and valleys. Suitably chosen profiles will produce high flutter factors and significant alternating-gradient focusing, enabling vertical focusing to be maintained up to 1 GeV or more using radial rather than spiral sectors.

INTRODUCTION
In an isochronous cyclotron, the constant orbit frequency, independent of ion energy \( \gamma m_0 c^2 \) and average radius \( R \) (circumference/2\( \pi \)), implies that

\[
B = \beta B_c, \quad R = \beta R_c, \quad (1)
\]

where \( B \) denotes the average field around a closed orbit, \( B_c \) the “central field” and \( R \), the “cyclotron radius”. Unfortunately the resultant positive field gradient produces a defocusing contribution to the vertical betatron tune \( \nu_z \) given by \( \Delta \nu_z^2 = -\beta^2 \gamma^2 \). From the beginning, a major problem in cyclotron design has been how to compensate this and ensure vertical focusing. Thomas’s [1] suggestion of edge focusing through an azimuthal field variation with \( N \)-fold symmetry, and Kerst’s [2] of adding alternating focusing by using spiral sectors, have together succeeded in enabling “compact” cyclotrons with spiral sectors to accelerate protons to 230 MeV (\( \beta^2 \gamma^2 = 0.55 \)) [3]. Separate-sector cyclotrons (SSCs) can achieve higher flutter and so higher energies: the PSI Ring Cyclotron [4] produces 590 MeV protons (\( \beta^2 \gamma^2 = 1.65 \)), and designs have been published for energies, up to 15 GeV [5].

Reverse-bend cyclotrons would achieve higher flutter still by making the valley fields negative (\( B_v = -B_h \), as in radial-sector FFAGs), rather than zero. Moreover, the alternating-gradient (AG) focusing, minimal in the previous schemes, becomes significant. Thus, if the hills cover a fraction of the orbit \( \eta = 0.6 \), the flutter is expected to maintain vertical focusing only up to 3.75 GeV. But a tracking simulation [6] has shown that positive focusing is in fact preserved up to 7.3 GeV.

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A common feature of the above schemes is that the hill and valley fields, while of different magnitudes, are assumed to have the same radial profiles, i.e.

\[
B_h(r)/B_v(r) = \text{constant} \quad (2)
\]

More elaborate designs have also been proposed to achieve isochronism at high energy without using spiral magnets – basically by introducing more free parameters. Thus Rees [7] has designed a non-scaling muon FFAG that remains isochronous over the range 8-20 GeV (5,900 < \( \beta^2 \gamma^2 < 37,000 \)), relying on AG focusing by “pumplet” cells (OdoFoDoFodO) composed of five straight-sided magnets of three different designs. On a less ambitious scale, Johnstone [8, 9] has designed close-to-isochronous non-scaling proton FFAGs to provide 250-MeV protons and 400-MeV carbon ions for cancer therapy, and 1-GeV protons for ADSR. These all use a 4-cell FDF triplet lattice with straight-sided (though not necessarily radial) edges, and are also remarkable for their low variation in tune, both \( \nu_z \) and \( \nu_r \). In both these authors’ studies the \( B(r) \) profile in each type of magnet is specially determined to produce the desired orbit properties.

Here we propose to explore a simpler possibility for achieving positive vertical focusing at high energy with purely radial sectors – allowing the radial field profiles in hills and valleys to differ. As was found helpful in previous high-energy cyclotron studies [6], we assume hard-edge fields with \( B_h \) and \( B_v \) each constant along equilibrium orbits. In particular we assume a polynomial variation with energy:

\[
B_h(\gamma) = H_0 + H_1 \gamma + H_2 \gamma^2 + H_3 \gamma^3 + \ldots \quad (3)
\]

\[
B_v(\gamma) = V_0 + V_1 \gamma + V_2 \gamma^2 + V_3 \gamma^3 + \ldots \quad (4)
\]

As a first step we consider a “compact” design with no drift spaces and negative valley fields. For an orbit of mean radius \( R \) crossing a hill-valley edge at radius \( R_e \), we may write \( \ell_h = \rho_h \psi_h \) and \( \ell_v = \rho_v \psi_v \) for the arc lengths within a half-cell (Fig. 1), where the radii of curvature \( \rho_h = B,R,\beta \gamma B(\gamma) \), \( \rho_v = B,R,\beta \gamma B(\gamma) \), and \( \psi_h \) and \( \psi_v \) are the bending angles.

Figure 1: Orbit geometry within a half-cell.
To maintain isochronism:
\[
\ell_\beta H + \ell_\gamma V_\beta = \frac{N}{N} \beta R \beta \quad \text{and} \quad \ell_\beta H + \ell_\gamma V_\gamma = 0 \quad (n \neq 1). \quad (5)
\]
Thus if the hill coefficients \(H_n\) are specified, the valley coefficients \(V_n\) must satisfy:
\[
V_\beta = \frac{\pi B R}{N} \beta - \frac{\ell_\gamma}{\ell_\gamma} H_\gamma \quad \text{and} \quad V_\gamma = -\frac{\ell_\beta}{\ell_\beta} H_\beta \quad (n \neq 1). \quad (6)
\]

**Orbit Geometry**

To compute \(B_c\) therefore requires knowledge of the bending angles \(\psi_\phi\) and \(\psi_\gamma\) and of the radii of curvature – of which \(\rho_\nu\) itself depends on \(B_c\). These parameters may be evaluated by invoking their various geometrical relationships, which after some manipulation yield a transcendental equation for \(\psi_\phi\), from which the others follow:
\[
\psi_\phi + \left(\psi_\phi - \pi/N\right) \frac{\sin \psi_\phi \sin[(1-h)\pi/N]}{\sin[\psi_\phi - \pi/N] \sin[h\pi/N]} = \frac{B_\beta}{B_\gamma} \psi_\gamma. \quad (7)
\]
This must be solved numerically, but a good starting point is to make the approximation \(K_\beta = \beta R\), giving:
\[
\psi_{\phi,0} = \arcsin \left( \frac{B_\beta}{B_\gamma} \sin \left( \frac{h\pi/N}{N} \right) \right). \quad (8)
\]

**Betatron Tunes**

To calculate the betatron tunes we take a lumped-element approach (validated by tracking with CYCLOPS in previous studies [6]), evaluating the traces of the vertical and horizontal transfer matrices for the full cell:
\[
M = M_\phi M_\rho M_\phi M_\rho, \quad (9)
\]
Here \(M_\phi\) is the standard \(2 \times 2\) matrix for a thin lens, while \(M_\rho\) and \(M_\phi\) are those for focusing and defocusing sector magnets respectively. For \(M_\rho\), we need to evaluate the focal power \(g\) of the edge crossing, which depends on the Thomas crossing angle \(\kappa = \psi_\rho - h\pi/N\) and is given by:
\[
g = \frac{B_\beta - B_\gamma}{B_\beta B_\gamma} \tan \left( \psi_\rho - \frac{h\pi}{N} \right). \quad (10)
\]
For \(M_\phi\) and \(M_\rho\) we need the field gradients and the phase advances \(\phi_n\). For vertical motion \(\phi = \ell_\gamma \sqrt{K_\gamma}\) and \(\phi = \ell_\gamma \sqrt{K_\gamma}\), where the respective coefficients \(K_\phi\) and \(K_\rho\) are:
\[
K_\phi = \frac{dB_\phi}{dr} / B_\phi R_\phi \beta = \frac{\gamma^2}{R_\phi^2} \left( H_1 + 2H_2 \gamma^2 + 3H_3 \gamma^3 + \ldots \right), \quad (11)
\]
\[
K_\rho = \frac{dB_\rho}{dr} / B_\rho R_\rho \beta = \frac{\gamma^2}{R_\rho^2} \left( V_\phi + 2V_\beta \gamma^2 + 3V_\gamma \gamma^3 + \ldots \right). \quad (12)
\]
For the horizontal motion the phase advances are \(\phi_n^* = \ell_\phi \sqrt{K_n^*}\) and \(\phi_n^* = \ell_\rho \sqrt{K_n^*}\), where:
\[
K_n^* = \frac{1}{\rho_n^*} + K_n \quad \text{and} \quad K_n^* = \frac{1}{\rho_n^*} - K_n. \quad (13)
\]

**RESULTS**

A number of cases were studied with \(H_0 = 0 = H_{\phi,0}\) to investigate the dependence of the tunes on the size of the \(H_1\) and \(H_2\) components, the hill fraction \(h\) and the number of sectors \(N\). Most runs were made for \(h = 0.5\) and \(N = 8\).

Figure 2 displays an example of such field profiles for one of the cases studied, showing that for \(H_2 = 0.2H_1\) the \(B_c\) required is 20-40% higher than in an SSC.

**Novel Cyclotrons and FFAGs**

No Sub Class

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**Figure 2:** Field profiles for \(N = 8\), \(h = 0.5\), \(H_1 = 2B_c\), \(H_2 = 0.4B_c\).

For reference Fig. 3 displays the tunes for the conventional situation where \(H_2 = 0\) so that both \(B_h\) and \(B_c\) are directly proportional to \(\gamma\). Results are shown for values of \(H_1/B_c\) between 2.0 and 2.6, the lower value representing the situation in a separated-sector cyclotron with \(h = 0.5\) and near-zero field in the valleys. In that case the flutter \(F^2 = 1\), and as expected the vertical tune \(\nu_v\) drops towards zero as \(\beta R \gamma^2\) approaches 1 at around 400 MeV. Higher values of \(H_1/B_c\) require more negative valley fields, raising the flutter and the vertical tune values, until with \(H_1/B_c = 2.6\) positive focusing is retained up to 1 GeV. The effect of increased \(H_1/B_c\) on the horizontal tune is less dramatic: for \(H_1/B_c = 2\), \(\nu_v = \gamma\), and for \(H_1/B_c = 2.6\) \(\nu_v\) still grows linearly but ~40% faster.

**Figure 3:** Variation of the vertical tune \(\nu_v\) with energy for \(H_2 = 0\), \(h = 0.5\), \(N = 8\), and various values of \(H_1/B_c\).

Figure 4 shows the effect of adding a \(\gamma^2\) component to the field for \(H_1/B_c = 2.0\), \(h = 0.5\) and \(N = 8\). For \(H_2 = 0\) we have the same case as before, where the vertical tune drops to 0 near 400 MeV. But increasing the \(\gamma^2\) component raises \(\nu_v\) significantly, making it almost constant for \(H_1/B_c = 0.4\) and rise with energy for higher values. There is also a significant effect on the horizontal tune, introducing a noticeable quadratic dependence on energy, driving \(\nu_v\) to the \(N/2\) resonance at 900 MeV for \(H_1/B_c = 0.6\).

In Fig. 5 we return to the effect of varying \(H_1/B_c\) (as in Fig. 2) for \(h = 0.5\) and \(N = 8\), but now with \(H_1/B_c = 0.4\). As before, raising \(H_1/B_c\) increases both tunes, but \(\nu_v\) more
The least variation in $\nu_z$ is obtained for $H_1/B_c = 2.2$. In this case (and in those of varying $h$ and $N$ below) the effect on the horizontal tune is small.

The fraction of a sector occupied by the hill also has a powerful influence on the tunes. Figure 6 shows the tunes for hill fractions $h$ between 0.5 and 0.65 for $H_1/B_c = 2$, $H_2/B_c = 0.4$ and $N = 8$. Just as for separated-sector cyclotrons, widening the hills increases the flutter and both $\nu_z$ and $\nu_r$.

The effect of the number of sectors on the tunes is shown in Fig. 7, where data are plotted for $N = 8, 10$ and 12, with $H_1/B_c = 2$, $H_2/B_c = 0.4$ and $h = 0.5$. Increasing $N$ lowers both $\nu_z$ and $\nu_r$, but the effect is a weak one except at the highest energies. Neither tune approaches the $N/2$ resonance in the energy range considered, but $\nu_r$ would do so for $N = 8$ not much above 1 GeV, making higher periodicity necessary for higher-energy designs.

**CONCLUSIONS**

A study has begun of using different radial field profiles in hills and valleys (while maintaining isochronism) to obtain increased flutter and more strongly alternating gradients – and hence increased vertical focusing - in radial-sector cyclotrons. As a first step, adding a $\gamma^2$ component to the hill fields in a “compact” design (i.e. no field-free regions) - and subtracting a compensating $\gamma^2$ component from the valley fields - has been shown to be a possible way of providing radial-sector cyclotrons with sufficient vertical focusing to reach at least 1 GeV.

The practicality of such a design has not been taken into account, particularly with regard to finding suitable locations for the accelerating cavities and injection and extraction systems. Field-free drift spaces would remove this difficulty and a beam optics study of such an arrangement is under way.

**REFERENCES**