CALCULATION OF COHERENT WIGGLER RADIATION USING EIGENFUNCTION EXPANSION METHOD

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Abstract

An analytic method originated by Y. H. Chin was extended to calculate the electromagnetic fields and the longitudinal impedance due to coherent wiggler radiation (CWR) in a rectangular chamber. The method used dyadic Green functions based on eigenfunction expansion method in electromagnetic theory and was rigorous for the case of straight chamber. We re-derived the theory and did find the full expressions for the longitudinal impedance of a wiggler with finite length. With shielding of chamber, the CWR impedance indicated resonant properties which were not seen in the theory for CWR in free space.

INTRODUCTION

In a damping ring where wigglers are used for main purpose of radiation damping, coherent radiation in the wigglers can contribute to beam coupling impedance. The impedance from undulator or wiggler was first studied in Refs. [1, 2]. Simple formula was found for an infinite long wiggler in free space [3] and applied to the instability analysis in a storage ring [4]. In this paper, the method described in Ref. [1] is extended to calculate the electromagnetic fields and the longitudinal impedance due to wiggler radiation in a rectangular chamber. Due to limit of space, we only outline the key schemes and present some main results. The interested readers are referred to Refs. [5, 6] for detailed derivations.

PROBLEM STATEMENT

Consider a straight rectangular pipe with infinite length and transverse dimensions of a and b, a wiggler is located at $0 \le z \le L$. The geometry of the problem is shown in Fig. 1. The source charge density is defined by Delta functions as

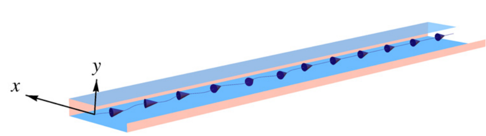

Figure 1: The geometry of the straight rectangular chamber for a wiggler. The beam moves along the curved line with arrows.

$$\varrho(\vec{r},t) = e\delta\left(x-x_t\right)\delta\left(y-y_t\right)\delta\left(z-z_t\right). \quad (1)$$

The orbit of the source particle, as shown in Fig. 2, is defined in the $x\!-\!z$ plane by simple sine function: $x_t(t) = (\theta_0/k_w)\sin(\omega_0 t) + x_0$ for $0 \le t \le L/v$, and $x_t(t) = x_0$ for $t < 0$ or $t > L/v$; $y_t(t) = y_0$ and $z_t(t) = vt$ for $-\infty < t < \infty$. Here we assume that $x_0 = a/2$, $y_0 = b/2$, $\theta_0 = K/\gamma$ and $k_w = \omega_0/c$. The quantity K is the wiggler strength parameter, γ is the relativistic factor, and k_w is the wavenumber of the wiggler. It is also assumed that $L = 2\pi N_w/k_w$ where N_w is integer, and the charge particle has constant velocity along z axis.

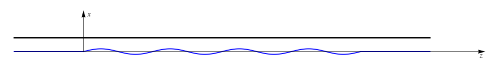

Figure 2: The beam orbit inside a wiggler. Note that the chamber width and orbit amplitude adopt different scale.

The current density is given by $\vec{\mathcal{J}} = \varrho\vec{v}$. The problem is to be solved by finding the solutions of two inhomogeneous Helmholtz equations for vector and scalar potentials in the frequency domain of

$$\nabla^2\vec{A} + k^2\vec{A} = -\mu_0\vec{J} \quad (2)$$

and

$$\nabla^2\Phi + k^2\Phi = -\frac{\rho}{\epsilon_0}, \quad (3)$$

under the Lorentz gauge condition of $\Phi = \frac{c^2}{i\omega}\nabla\cdot\vec{A}$. Here we define $k \equiv \omega/c$, and the quantities $\vec{J}$ and ρ are respectively the Fourier transforms of $\vec{\mathcal{J}}$ and ϱ. The electric field is given by

$$\vec{E} = i\omega\vec{A} - \nabla\Phi = i\omega\vec{A} - \frac{c^2}{i\omega}\nabla\nabla\cdot\vec{A}. \quad (4)$$

With perfectly conducting walls, the boundary condition is $\vec{n}\times\vec{E} = 0$, and $\vec{n}$ is the unit vector normal to the surface. Assume that a test particle adopts the same orbit as the source particle does, but with a time delay τ. The longitudinal wake can be calculated as the work of the electric field done on the test particle. In general, τ can be positive or negative values, indicating the test particle is behind or ahead of the source particle. The wake function is

$$W(\tau) = -\frac{1}{e^2}\int d\vec{r}\int_{\tau}^{\tau+\frac{L}{v}} dt\,\vec{\mathcal{J}}\left(\vec{r},t-\tau\right)\cdot\vec{\mathcal{E}}\left(\vec{r},t\right). \quad (5)$$

The longitudinal impedance is related to the wake function via Fourier transform as follows

$$W\left(\tau\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z\left(\omega\right)e^{-i\omega\tau}d\omega. \quad (6)$$

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EIGENFUNCTION EXPANSION METHOD

The eigenfunction expansion method (also called Ohm-Rayleigh method in the literature) can be used to solve ordinary or partial differential equations (PDE) with the boundary conditions are separable to each of the variables. Suppose an inhomogeneous differential equation for the distribution function $y(\vec{r})$ reads $\mathcal{L}y(\vec{r}) = -f(\vec{r})$, where $\mathcal{L}$ is a linear differential operator and $f(\vec{r})$ represents the source distribution. The Green's function is taken to be the solution of $\mathcal{L}G(\vec{r},\vec{r}') = -\delta(\vec{r}-\vec{r}')$, where $\delta(z)$ is the Dirac delta function. If $\vec{r} = (x, y, z)$, then $\delta(\vec{r}) = \delta(x)\delta(y)\delta(z)$. An important property of Green's function is the symmetry of its two variables; that is, $G(\vec{r},\vec{r}') = G(\vec{r}',\vec{r})$.

In a bounded region, the Green's function can be uniquely determined by applying boundary conditions or/and initial conditions. The region of the problem can be infinite extent or a bounding surface. In terms of Green's function, the particular solution of $y(\vec{r})$ takes on the form $y(\vec{r}) = \int_V G(\vec{r},\vec{r}')f(\vec{r}')\,\mathrm{d}V$. It indicates that the Green's function enters in an integral solution of the original PDE. Therefore, the problem of solving the PDE changes to solving the equation of Green's function.

To find the solution of Eq. (2), the Green's functions relevant to the three components of $\vec{A}$ satisfy

$$\nabla^2 G_\nu(\vec{r},\vec{r}') + k^2 G_\nu(\vec{r},\vec{r}') = -\delta(\vec{r}-\vec{r}'), \quad (7)$$

where $\nu = x,\ y,$ or z and $\delta(\vec{r}-\vec{r}') = \delta(x-x')\delta(y-y')\delta(z-z')$. For a passive waveguide, the delta function of z in Eq. (7) can be replaced by its Fourier transform in the form of

$$\delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\beta(z-z')}\,d\beta. \quad (8)$$

The delta function of transverse coordinates can be expanded into the summation of the eigenmodes of the rectangular waveguide as follows

$$\delta(\vec{r}_\perp - \vec{r}'_\perp) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\phi_\nu(\vec{r}_\perp)\,\phi_\nu(\vec{r}'_\perp), \quad (9)$$

where $\nu = x,\ y,$ or z, and the subscript $\perp$ denotes the transverse coordinates. And the complete set of orthonormal eigenfunction for the x, y and z directions are

$$\phi_x(\vec{r}_\perp) = \frac{2}{\sqrt{(1+\delta_{m0})\,ab}}\cos(k_x x)\sin(k_y y), \quad (10)$$

$$\phi_y(\vec{r}_\perp) = \frac{2}{\sqrt{(1+\delta_{n0})\,ab}}\sin(k_x x)\cos(k_y y), \quad (11)$$

$$\phi_z(\vec{r}_\perp) = \frac{2}{\sqrt{ab}}\sin(k_x x)\sin(k_y y), \quad (12)$$

where $k_x = m\pi/a$ and $k_y = n\pi/b$ are the transverse wave numbers. Finally, the solution of Eq. (7) can be obtained as [7, 1]

$$G_\nu = \frac{1}{2\pi}\sum_{m,n\geq 0}\int_{-\infty}^{\infty} d\beta\,\frac{\phi_\nu(\vec{r}_\perp)\,\phi_\nu(\vec{r}'_\perp)}{\beta^2 - k_z^2}e^{i\beta(z-z')}, \quad (13)$$

where $k_z^2 \equiv k^2 - (k_x^2 + k_y^2)$. Note that when writing down the solution of Green's function, the boundary condition has been applied. With in hand the explicit form of G_ν, one is able to find the solution of the vector potential as $A_\nu(\vec{r},\omega) = \mu_0 \int d\vec{r}'\, G_\nu(\vec{r},\vec{r}')\, J_\nu(\vec{r}',\omega)$.

The integral in terms of β in Eq. (13) should be evaluated by means of complex integration in the β-plane. The term of $\beta^2 - k_z^2$ in the denominator determine the poles of the integral. There are ambiguities associated with the singularity of the Green's function in the source region. They are essential and need to be clarified in order to obtain the full solution of the wiggler radiation problem. These issues are put forward in detail in Refs. [5, 6].

SELECTED RESULTS

Beam spectrum

We first calculate the beam spectrum of the current density using the eigenfunction expansion method. With the help of Jacobi-Anger [7] expansion, the results are

$$J_x(\vec{r},\omega) = \frac{eck_w}{2v}\sum_{m,n\geq 0}\sum_{p=-\infty}^{\infty}\frac{4pF_{mnp}\phi'_x(x,y)}{(1+\delta_{m0})abk_x}e^{i\beta_p z}, \quad (14)$$

$$J_z(\vec{r},\omega) = \frac{e}{2i}\sum_{m,n\geq 0}\sum_{p=-\infty}^{\infty}\frac{4F_{mnp}\phi'_z(x,y)}{ab}e^{i\beta_p z}, \quad (15)$$

where we define $\phi'_x(x,y) = \cos(k_x x)\sin(k_y y)$, $\phi'_z(x,y) = \sin(k_x x)\sin(k_y y)$, $\beta_p = (\omega + p\omega_0)/v$, and

$$F_{mnp}(x_0,y_0) = \sin(k_y y_0)C_x J_p(k_x\frac{\theta_0}{k_w}) \quad (16)$$

with $C_x = e^{ik_x x_0} - (-1)^p e^{-ik_x x_0}$. The quantity J_p represents the p-th order Bessel's function. It is obvious that $J_y = 0$ for a planar wiggler.

It is seen that the beam spectrum contains harmonics at orders up to infinity. Consequently, the fields excited by a point charge moving along a wiggler can be interpreted by the dispersion relation as shown in Fig. 3. In the figure, the dispersion relation of a rectangular waveguide is given by $k^2 = k_x^2 + k_y^2 + k_z^2$, and the beam mode by $\beta_p = k_z$. The beam modes with $p \neq 0$ are shifted due to the wiggling motion. As sketched in the figure, the waveguide dispersion relation is extended to include slow waves, which corresponds to imaginary values of k and k_z. The upshifted beam modes can couple with the propagating waveguide modes. This mode synchronism leads to the well-known wiggler radiation. On the other hand, the downshifted beam modes can couple with the slow waves. It leads to a kind of radiative fields similar to space-charge fields. The crossing points between the beam modes and the fast/slow waves indicate real/imaginary poles in the complex wavenumber plane. The values of these poles are then used in evaluating the complex integration of Eq. (13).

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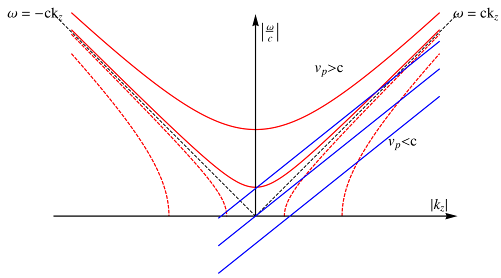

Figure 3: Dispersion relation for the waves excited by a point charge moving along a perfectly conducting waveguide sandwiched by a wiggler. The solid blue line denotes the beam mode.

Impedance due to real poles

Since the general formulae for the impedance are very complicated, we only present simplified results for the first-order harmonic. The particle velocity is assumed to be $v = c$, and this is certainly true at high energy electron or positron rings. Another assumption is that the deflecting angle is small, i.e. $\theta_0 \ll 1$. Then the longitudinal impedance due to real poles is given by

$$Z_{\parallel}^{1}(k) = Z_0\theta_0^2\Theta_1(k), \tag{17}$$

where the dimensionless function $\Theta_1(k)$ is

$$\Theta_1(k) = \frac{L^2}{ab}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{k}{(1+\delta_{m0})k_z}\cos^2(k_x x_0)\sin^2(k_y y_0) \times \left\{ \left(\frac{k-k_z}{k-k_z+k_w}\right)^2\left[S^+(\alpha^-)+\frac{i}{(k-k_z)L}\right] + \left(\frac{k+k_z}{k+k_z+k_w}\right)^2\left[S^-(\alpha^+)-\frac{i}{(k+k_z)L}\right]\right\} \tag{18}$$

with $\alpha^- = (k-k_z-k_w)L/2$ and $\alpha^+ = (k+k_z-k_w)L/2$. Here two sink-like functions are defined

$$S^+(z) = \frac{\sin^2(z) + i\left[\sin(z)\cos(z) - z\right]}{z^2}, \tag{19}$$

and

$$S^-(z) = \frac{\sin^2(z) - i\left[\sin(z)\cos(z) - z\right]}{z^2}. \tag{20}$$

The profile of the impedance is mainly determined by the sink-like functions $S^+(z)$ and $S^-(z)$. That is, the wiggler radiation impedance is high peaked around resonances at $k - k_w - k_z = 0$ when N_w is large. On the other hand, the real and imaginary parts of Eq. (17) satisfy the Kramers-Kronig relation. Therefore, the wake function corresponding to $Z_{\parallel}^{1}(k)$ is a causal function.

Impedance due to imaginary poles

The impedance due to the imaginary poles is of importance in the wiggler radiation theory. The idea of considering imaginary poles was originated in Ref. [8]. The impedance due to imaginary poles is related to space-charge (or beam self-fields) effect. As did in Ref. [8], the trick is to do replacement $k \to ik$, and then re-derive the whole theory. It was found that the CWR impedance due to imaginary poles can be described by

$$\bar{Z}_{\parallel}^{1}(k) = iZ_{\parallel}^{1}(ik). \tag{21}$$

The simplified form of $\bar{Z}_{\parallel}^{1}(k)$ turns out to be

$$\bar{Z}_{\parallel}^{1}(k) = Z_0\theta_0^2\bar{\Theta}_1(k), \tag{22}$$

where

$$\bar{\Theta}_1(k) = \frac{i}{ab}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{k}{(1+\delta_{m0})\bar{k}_z}\cos^2(k_x x_0)\sin^2(k_y y_0) \times \left\{ \frac{k_-\left[L\left(k_-^2+k_w^2\right)+2k_-(e^{-k_-L}-1)\right]}{\left(k_-^2+k_w^2\right)^2} + \frac{k_+\left[L\left(k_+^2+k_w^2\right)+2k_+(e^{-k_+L}-1)\right]}{\left(k_+^2+k_w^2\right)^2}\right\} \tag{23}$$

with $\bar{k}_z = \sqrt{k^2+k_x^2+k_y^2}$, $k_+ = \bar{k}_z + k$ and $k_- = \bar{k}_z - k$. In the above equation, the second term in the curly braces is small comparing with the first term. It is obvious that $\bar{Z}_{\parallel}^{1}(k)$ is purely imaginary and has not resonant peaks as shown in $Z_{\parallel}^{1}(k)$. It is more similar to the impedance due to the imaginary poles in the coherent synchrotron radiation theory [8]. This kind of impedance share the properties of space-charge and is intimately related to the overtaking fields which always cling to the beam itself.

SUMMARY

The theory of eigenfunction expansion method is re-derived and made suitable for calculating the full expressions for wiggler radiation fields and impedance. As an improvement of the original theory in Ref. [1], the imaginary part of the longitudinal wiggler radiation impedance has been successfully obtained. When Fourier transform is used to solve the system in frequency domain, the imaginary frequency has to be introduced as a mathematical tool to include the anomalous part of the charge self-fields.

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