A MATRIX PRESENTATION FOR A BEAM PROPAGATOR INCLUDING PARTICLES SPIN

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Abstract

This approach based on the matrix formalism for Lie algebraic tools provides a constructive method for a beam propagator in magnetic and electrical fields. The beam propagator is evaluated in accordance to the well known Lie algebraic tools. But in contrast to traditional approaches matrix presentation for Lie propagators bases on two dimensional matrices.

INTRODUCTION

High-energy and nuclear physics using accelerators has reached a point where a very large fraction of the experiments require polarized beams. One of the the greatest triumph is the recent successful installation and commissioning of 'Siberian Snakes' and spin rotators at RHIC, the Relativistic Heavy-Ion Collider at BNL. RHIC is the world’s first polarized proton collider. There are several ongoing works for creation different types of accelerators with polarized beams usage (for example NICA machine JINR, Dubna, Russia). The spin program is an important and integral part also for the NICA project. Indeed, ever since the “spin crisis” of 1987, the composition of the nucleons spin in terms of the fundamental constituents - quarks and gluons - remains in the focus of attention of many physicists. This section contains the discussion of the physics goals and perspectives of the spin program at NICA. The highlights of the NICA spin program include the measurements of Drell-Yan processes with longitudinally polarized proton and deuteron beams, spin effects in the inclusive and exclusive production of baryons, light and heavy mesons and direct photons, and the studies of helicity amplitudes and double spin asymmetries in elastic scattering. This section also addresses the issue of the competitiveness of the NICA spin program - it appears that the SPD detector at NICA would allow to contribute significantly to the current and planned international program in spin physics.

EQUATIONS

In general case particle motion equations

\[
dX/dt = F(X, t) = \sum_{k=1}^{\infty} \mathbb{R}^{1k} X^{[k]}
\]

(1)

can be presented using so called matrix formalism for differential equations [1]. The solution of Eq. 1 can be written in the following form

\[
X(t) = \sum_{k=1}^{\infty} \mathbb{R}^{1k} (t|t_0) X_0^{[k]}, \quad X_0 = X(t_0).
\]

where \(X_0^{[k]}\) is so called Kronecker power of \(k\)-the order of the initial phase vector \(X_0\).

The matrices \(\mathbb{R}^{1k} (t|t_0)\) can be evaluated using the matrix formalism for Lie algebraic tools [1]. In linear approximation one obtains well known linear solution

\[
X = \mathbb{R}^{11} (t|t_0) X_0, \quad X_0 = X(t_0).
\]

The vector of spin components \(S\) changes with the time \(t\) of the laboratory frame according to the Thomas-Bargmann-Michael-Teledi (T-BMT) equation (see i.e. [2]). The spin precessions equation can be written in two following forms

\[
dS/dt = W_s \times S = \mathbb{W} \cdot S,
\]

where \(\mathbb{W}\) is a skew-symmetric matrix

\[
W_s = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \quad \text{and} \quad \mathbb{W} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}.
\]

We have to understand that \(W_s\) depends of the particle position in phase space \(W_s = W_s(X)\) and lose some advantages of matrix formalism representation. However we can courageously build it in linear approach. In introduction we already mentioned about EDM-machine which will have only electrostatic elements. But in this paper neglect the electric fields (do’nt generality) and leave only components of magnetic field \(B\). It is used to work in accelerator coordinate system and all differentiation are by the independent variable \(s\) measured along a reference orbit. The process of converting the spin precession equation to accelerator coordinates described as well in [2, 3]. In this coordinate system one can write

\[
W_s = -\frac{e}{p c} (1 + h x) \left[ \alpha \gamma + 1 \right] \left( B_x e_1 + B_y e_2 + B_y e_3 \right) - \frac{\alpha \gamma^2 \beta^2 B_x x' + B_s (1 + h) + B_y y'}{\gamma + 1} \left( 1 + h x \right)^2 + x'^2 + y'^2 \times \left( x' e_1 + (1 + h x) e_2 + y' e_3 \right)
\]

(2)

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This expression can be used for evaluation of components for the matrix \( W \). For example, for the element \( w_1 \) one can write
\[
 w_1 = -\frac{e}{pc}(1 + hx)(a\gamma + 1)B_x - \frac{\alpha^2 \beta^2}{\gamma + 1}(1 + hx)^2 + B_x(1 + h) + B_y y' + y'^2 \gamma^2 \]
(3)

The equalities (2) and (4) are used for forming our skew-symmetric matrix \( W \).

For the solution of this system we realize the next two steps.

At first, according to the paper [4] we should introduce the ensemble-average spin vector \( \langle \vec{S}(X, s) \rangle_T \) in accord according to the following equality
\[
 \langle \vec{S}(X, s) \rangle_T = \lim_{T \to \infty} \frac{1}{T} \int_0^T \vec{S}(X, s) ds,
\]
(5)

For the next we also introduce the ensemble-average spin vector in accord with the following equality
\[
 \langle \vec{S}(X, s) \rangle_M = \frac{1}{mes(\Omega)} \int_{\Omega} \vec{S}(X, s) dX,
\]
(6)

where \( \Omega \) is the space phase set occupied by beam particles and \( mes(\Omega) \) is its measure. It is not difficult to show that our dynamical system (4) is an ergodic system. It is known that for similar system it is true the following equality
\[
 \langle \vec{S}(X, s) \rangle_T = \langle \vec{S}(X, s) \rangle_M.
\]
(7)

This equality allows us to use the approach described in the book [1].

In accord with this approach one should change the right part of the second equation of Eq. 4 by the following equation:
\[
 \frac{d(S)_{\Omega}(s)}{ds} = \langle W \rangle_{\Omega} \cdot (S)_{\Omega}(s), \tag{6}
\]

where \( (S(s))_{\Omega} \) means that distribution function of particles spin includes in \( F_{spin} \) by integral way and integrating are by \( \Omega = M(s) \).

Let us introduce an operator and a function in accord with the following rules: an evolution operator
\[
 M(s|s_0) : S(s_0) \to S(s)
\]
and the function
\[
 F_{spin} = \langle W \rangle_{\Omega} \cdot (S)_{\Omega}(s).
\]

These operator and function allows us to write the following equation
\[
 \frac{dM(s, s_0, V)}{ds} = \mathcal{V} \circ M(s, s_0, V),
\]
where
\[
 \mathcal{V} = F_{spin} \frac{\partial}{\partial s}.
\]

and initial state condition is
\[
 M(s, s_0, V) = I d,
\]
where \( I d \) is an identity operator. In accord with [1] we can write the following algorithm for solution of our problem

It can be matched an integral equation in Volterr-Urisone form. Write this equation in formal form
\[
 M = A \circ M, \tag{7}
\]
where \( A \) – Urison’s operator. The main seal of solution existence of Eq. 7 is the method of successive approximations which helps to find out the existence of stable point of operator \( A \). In other words has to be build a sequence \( M^k = A \circ M^{k - 1} \) by some initial value \( M^0 \) then proof a convergence of \( M^k \) to some \( M^* \) and there is the parity \( M^* = A \circ M^* \).

Let consider the general aspects of building a solution for Eq. 7 for motion with spin adjusted system (Eq. 4).

- **Step 0.** First of all set an interval for a solution \([s_0, s_1]\), \( \Delta s = s_1 - s_0 \). For the interval \([s_0, s_1]\) define a system of transportation therefore \( X \) – a component of \( F_{spin} \) function.

- **Step 1.** Then choose the distribution function \( S(X(t), s_0) = S_0(X) \) see [1, 4].

- **Step 2.** Calculate the evolution operator
\[
 M : M^0 = M(s|s_0, V), s \in [s_0, s_1].
\]
Step 3. Evaluate the current value of distribution function $S^1(X, s) = S_0((M^0)^{-1} \circ X_0)$.

Step 4. Solve corresponding spin equations with distributing function $S^1(X, s)$ and retrieve $S^1$.

Step 5. Evaluate function $F_{\text{spin}} = F_{\text{spin}}(S, X, s)$.

Step 6. One can solve the Eq. 7 and define $M^1 = A \circ M^0$.

Step 7. Retrieve a new value for $\langle f(X, t_0) \rangle^1_{\text{M}^1}$ uses formula

$$\langle S(X, s_0) \rangle^1_{\text{M}^1} = (1 - \alpha)\langle S_0((M^{'})^{-1} \circ X_0) \rangle^{\text{M}_0} +$$

$$+ \alpha \langle S_0((M^{\infty})^{-1} \circ X_0) \rangle^{\text{M}_0} \quad (8)$$

and $0 < \alpha < 1$.

Step 8. Checking the specific criterion

$$||M^k - A \circ M^{k-1}|| < \varepsilon, \quad k \geq 1. \quad (9)$$

If Eq. 9 is true the process of determination $M$ on the interval $[t_0, t_1]$ finished. But if the condition of (9) is false the process repeating from step four with corresponding functions and operators redefinition.

Treat the Eq. 8 in general form matches for any step

$$\langle S(X, s_0) \rangle^k_{\text{M}^k} =$$

$$= (1 - \alpha)\langle S_0((M^{'})^{-1} \circ X_0) \rangle^{\text{M}_0} +$$

$$+ \alpha \langle S_0((M^{\infty})^{-1} \circ X_0) \rangle^{\text{M}_0}, \quad (10)$$

and $0 < \alpha < 1$, $M^k = A \circ M^{k-1}$. This algorithm is different from some authors uses. Furthermore in most papers there are no studying a convergence problem except numerical algorithm examination.

**CONCLUSION**

This approach permit to apply all of matrix algebra opportunities and advantages in contrast with the tensor presentation based on multi-indexes description. The necessary computation can be realized in symbolic (using computer algebra codes as Mathematica, Mapple, Maxima and so on). The corresponding symbolic objects itself can be stored in special databases and used then in numerical computing. Parallel and distributed conception is well acceptable with the suggested matrix formalism.

**REFERENCES**


