EFFECT OF COULOMB COLLISIONS ON ECHO-ENABLED HARMONIC GENERATION (EEHG)*

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Abstract

We calculate the EEHG bunching factor with account of the collisions and derive a simple scaling relation for the strength of the effect. Our estimates show that collisions become a limiting factor in EEHG seeding for large harmonic numbers.

INTRODUCTION

Echo-enabled harmonic generation (EEHG) for FEL seeding uses two undulator-modulators and two chicanes to introduce a fine structure into the beam longitudinal phase space which, at the end of the system, transforms into a high harmonic modulation of the beam current [1, 2]. As a result of this phase space manipulation the energy distribution function after the first chicane becomes a rapidly modulated function of energy, with the scale of the modulation of the order of the initial energy spread of the beam divided by the EEHG harmonic number. Small-angle Coulomb collisions between the particles of the beam (also known as intra-beam scattering) tend to smear out this modulation and hence to suppress the beam bunching. In this paper we calculate the EEHG bunching factor with account of the collisions and derive a simple scaling relation for the strength of the effect.

It is well known that the dominant process in Coulomb collisions is a small-angle scattering, which leads to a diffusion process in the momentum space. We first derive the diffusion coefficient for this process in the beam frame using an approximation that the longitudinal temperature of the beam is much smaller than the transverse one. We then make a Lorentz transformation into the laboratory frame, and calculate the effect of the Coulomb collisions on the bunching factor in the EEHG seeding.

LORENTZ TRANSFORMATION OF A GAUSSIAN DISTRIBUTION FUNCTION

Consider a relativistic beam with the nominal energy \( E_0 = \gamma m_e c^2 \) \( (\gamma \gg 1) \) moving along a straight path in \( z \) direction. Assuming a Gaussian distribution function of the beam in the lab frame, we write it as follows

\[
f(p_x, p_y, p_z) = \frac{n_0}{(2\pi)^{3/2} p_0^2 \sigma^3 \sigma_{pc}} \times e^{-p_x^2/2p_0^2} e^{-p_y^2/2\sigma^2} e^{-p_z^2/2\sigma_{pc}^2},
\]

where \( n_0 \) is the beam density (number of particles per unit volume), \( p_x \) and \( p_y \) are the transverse components of the momenta, \( \sigma \) is the rms angular spread of the beam, \( p_0 = E_0/c \) is the nominal momentum, and \( \sigma_{pc} \) is the rms spread of the longitudinal momentum in the beam. The rms spreads of the transverse components of the momentum \( \sigma_{px} \) and \( \sigma_{py} \) are assumed equal to each other and are written as \( p_0 \sigma_\theta \), where \( \sigma_\theta \) is the rms angular spread of the beam. Note that in the limit when \( \sigma_{pc}/p_0 \gg \sigma_\theta^2 \), which we will assume here (see numerical estimation at the end of this section), one can identify \( \sigma_{pc} \) with the rms energy spread in the beam \( \sigma_E \). In what follows we will use the relative energy \( \eta = (E - E_0)/E_0 \), and correspondingly the rms spread \( \sigma_\eta = \sigma_E/E_0 \) and replace \( \sigma_{pc} \) in the distribution function (1) by \( p_0 \sigma_\theta \).

In order to obtain the distribution function \( F \) in the beam frame (moving with velocity \( p_0/c \) along \( z \)) one needs to use the Lorentz transformation for the momenta, and also take into account that the particle density in the beam frame is \( \gamma \) times smaller than in the lab frame. In addition, we will assume that particles’ velocities in the beam frame are non-relativistic. To simplify equations in what follows we will use the same notations \( p_x, p_y \), and \( p_z \) for the momenta components in the beam frame. A simple calculation gives

\[
F(p_x, p_y, p_z) = \frac{N}{(2\pi)^{3/2} T_\parallel T_\perp} \times e^{-p_x^2/2p_0^2} e^{-p_y^2/2\sigma^2} e^{-p_z^2/2\sigma_{pc}^2},
\]

where \( N = n_0/\gamma \) is the particle density in the beam frame, \( T_\parallel = m_e c^2 \sigma_\eta^2 \), \( T_\perp = m_e c^2 \sigma_\theta^2 \) (here \( n_0 \), \( \sigma_\theta \) and \( \sigma_\eta \) refer to the lab frame). The functions (1) and (2) expressed in the same variables are actually equal to each other in agreement with the fact that the distribution function is invariant with respect to the Lorentz transformation [3]. Our assumption of non-relativistic motion in the beam frame means \( T_\parallel, T_\perp < m_e c^2 \).

To illustrate the practicality of our assumptions, let us estimate the transverse and longitudinal temperatures of the beam with the normalized emittance \( \epsilon = 1 \, \mu m \), beam energy 1 GeV, \( \gamma \approx 2000 \), and the energy spread \( \sigma_\eta = 10^{-4} \) typical parameters for a soft x-ray FEL beam. Assuming that the beam is transported through a beam line with the beta function of 10 m, we find the angular spread \( \sigma_\theta = \sqrt{\epsilon/\gamma^2} \approx 7 \times 10^{-6} \) and \( T_\perp \approx 100 \) eV. For the longitudinal temperature one finds \( T_\parallel \approx 5 \times 10^{-3} \) eV. We see

\[
1 \text{In the opposite limit, } \sigma_{pc}/p_0 \ll \sigma_\theta^2, \text{ one cannot neglect } p_x \text{ and } p_y \text{ in the equation for energy, } E \approx p_c (1 + p_x^2/p_0^2 + p_y^2/p_0^2), \text{ and the angular spread of the beam is coupled to the energy spread.}
\]

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that non-relativistic condition $T_{∥}, T_{⊥} \ll m_e c^2$ is well satisfied. Moreover, we also see that $T_{∥} \ll T_{⊥}$. We will use this condition in the next section to simplify the Coulomb collision term.

**SIMPLIFIED COULOMB COLLISION TERM**

We saw in the previous section, that for typical parameters of an FEL beam $\sigma_∥, \gamma \sigma_∥$, and the longitudinal temperature of the beam in the beam frame is much smaller than the transverse one with both being non-relativistic. The collision term can be simplified in this limit using an approach similar to that developed in [4].

The EEHG distribution function of the beam after the first chicane can be represented as a product of a Gaussian distribution in the transverse direction, and a rapidly modulated Gaussian over the energy [5]. We approximate this function (in the beam reference frame) by

$$F(p, t) = F_2(p_x, p_y)F_1(p_z, t),$$

where $F_1$ is a one-dimensional distribution function over $p_z$ and

$$F_2(p_x, p_y) = \frac{1}{2\pi mT_{⊥}} e^{-\frac{(p_x^2 + p_y^2)}{2mT_{⊥}}}.$$  \hfill (4)

The function $F_1$ is normalized by $\int dp_z F_1 = N$. The characteristic width of $F_1$ is of order of $\sqrt{mT_{∥}}$, and, as mentioned above, it is modulated on the scale $\sim \sqrt{mT_{∥}/m}$, where $m$ is the harmonic number. The time dependence of $F_1$ indicates its evolution due to Coulomb collisions, however we neglect the time variation of $F_2$ because it is a much slower process compared with that of $F_1$.

Due to the intrabeam scattering $F$ will evolve in time and this evolution is described by the Langaus collision term [6]. The collision term involves first and second derivatives of $F$ with respect to momenta, of which the dominant one in our case will be $\partial^2 F/\partial p_z^2$, due to a rapid variation of $F$ along $p_z$. Keeping only this term allows us to write the kinetic equation for $F$ as

$$\frac{\partial F}{\partial t} = \frac{1}{2} D_{zz} \frac{\partial^2 F}{\partial p_z^2},$$

where the diffusion coefficient $D_{zz}$ is given by

$$D_{zz} = 4\pi N m_e e^4 \Lambda \psi,$$

with

$$\psi(p) = \int d^3 p' F(p') \frac{(p_x - p'_x)^2 + (p_y - p'_y)^2}{|p - p'|^3},$$

and $\Lambda$ the Coulomb logarithm (cf. [4]). Because $\psi$ is obtained by integration of $F$, the rapid energy modulation of $F$ over $p_z$ will be averaged out, and we can use (2) in evaluation of the integral (7). In the limit $T_{∥} \ll T_{⊥}$ one can also approximate $|p - p'| \approx ((p_x - p'_x)^2 + (p_y - p'_y)^2)^{1/2}$ and carry out integration over $p'_z$ in (7) using the normalization of $F_1$:

$$\psi(p_x, p_y) = N \int d^2 p' F_2(p'_x, p'_y) \times ((p_x - p'_x)^2 + (p_y - p'_y)^2)^{-1/2}. \hfill (8)$$

Note that in our approximation $\psi$ does not depend on $p_z$ and is also independent of time.

To calculate the integral in (8) we use the identity $R^{-1} = \sqrt{2\pi} \int_0^{\infty} d\xi e^{-\xi R^2}$, and rewrite (8) as

$$\psi(p_x, p_y) = \sqrt{\frac{2}{\pi}} N \int_0^{\infty} d\xi \int d^2 p' \times e^{-\xi ((p_x - p'_x)^2 + (p_y - p'_y)^2)/2} F_2(p'_x, p'_y).$$

The integration over $p_x$ and $p_y$ can now be easily carried out

$$\psi(p_x, p_y) = \frac{N}{\sqrt{2\pi m T_{⊥}}} \int_0^{\infty} d\xi \frac{d\xi}{(\xi + 1)^{3/2}} \times e^{-\xi (p_x^2 + p_y^2)/(2mT_{⊥}(\xi + 1))},$$

where we have introduced the new integration variable $\xi = \xi m T_{⊥}$.

To obtain an equation for $F_1$ we integrate (5) over $p_x$ and $p_y$:

$$\frac{\partial F_1}{\partial t} = \frac{1}{2} (D_{zz}) \frac{\partial^2 F_1}{\partial p_z^2},$$

where for $\langle D_{zz} \rangle = \int dp_x dp_y D_{zz}F_2(p_x, p_y)$ with the help of (4) and (10) one finds

$$\langle D_{zz} \rangle = 4\pi N m_e e^4 \Lambda \int dp_x dp_y \psi(p_x, p_y) \int dp_z d^2 p' F_2(p'_x, p'_y)$$

$$\times e^{-\xi (p_x^2 + p_y^2)/(2mT_{⊥}(\xi + 1))}.$$  \hfill (12)

This our result agrees with calculations of Ref. [4].

**TRANSFORMATION TO THE LABORATORY FRAME**

We will now express all quantities in the beam frame from the previous section in terms of the beam parameters in the lab frame. We will also average our diffusion equation over the transverse geometrical cross section of the beam.

The time in the beam frame is related to the distance $s$ traveled in the lab frame via the transformation $t \rightarrow s/c\gamma$. The momentum $p_z$ in the beam frame can be expressed through the energy deviation $\Delta E$ in the lab frame as $p_z \rightarrow \Delta E/\gamma c$. This converts (11) to the lab frame

$$\frac{\partial f}{\partial s} = \frac{\gamma c}{2} \langle D_{zz} \rangle \frac{\partial^2 f}{\partial \Delta E^2}.$$  \hfill (13)
where $f$ is the energy distribution function in the lab frame. Recalling the definitions $N = n/\gamma$, and $T_{\perp} = \mu y^2 \sigma_{\perp}^2$ and introducing the diffusion coefficient $D$ in the lab frame as

$$D = \gamma c \langle \Delta z^2 \rangle = \frac{2 \sqrt{3} \pi e^4 \Lambda}{\gamma \sigma_0},$$

(14)

we arrive at

$$\frac{\partial f}{\partial s} = \frac{1}{2} D \frac{\partial^2 f}{\partial \Delta E^2}.$$  

(15)

In our derivation we tacitly assumed that the beam density $n$ is a constant. In reality, it is a function of $x$, $y$ and $z$ which makes $D$ and $f$ also dependent on these coordinates. Since in EEHG we are interested in one-dimensional dynamics in $z$ direction we will average (15) over the transverse cross section of the beam assuming that the dependence of $n$ and $f$ versus $x$ and $y$ is given by $\exp[-(x^2 + y^2)/2\sigma_{\perp}^2]$ with $\sigma_{\perp}$ the rms transverse size of the beam. The result of the integration does not change the functional form of equation (15), but replaces the beam density $n$ in (14) by the following expression

$$n \rightarrow \frac{1}{4 \pi \sigma_{\perp}^2} \frac{l}{r_e I_A},$$

(16)

where $l$ is the beam current, $I_A = mc^3/e \approx 17$ kA is the Alfven current and $r_e$ is the classical electron radius. After the averaging the distribution function $f = f(z, \Delta E, s)$ remains dependent on $\Delta E$, $s$ and $z$. Finally, using the normalized transverse emittance $\varepsilon = \gamma \sigma_0 \sigma_{\perp}$, we obtain for the diffusion coefficient

$$D = \frac{\pi^{1/2} e^4 \Lambda}{2 \sigma_{\perp} \varepsilon} \frac{l}{r_e I_A}.$$  

(17)

In practical units, assuming $\Lambda \approx 8$ [7]

$$D = 3.1 \frac{l \text{ } [\text{kA}]}{(\sigma_x [\mu m]) (\sigma_y [100 \mu m])} \text{ keV}^2 \text{ m}.$$  

(18)

**APPLICATION FOR EEHG**

In application to EEHG we will adopt a model, in which we take collisions as occurring in a drift section of length $l$ after the first chicane. The justification of this approach lies in the fact that in this region the distribution function, being modulated in energy, is most sensitive to the collisions. This energy modulation actually persists through the system, and use for $l$ the combined lengths of the drift after the first chicane, the second undulator and the second chicane.

In what follows, we use notations of Ref. [2] with $A_1 = \Delta E_1/\sigma_E$ and $A_2 = \Delta E_2/\sigma_E$ for dimensionless amplitudes of the energy modulation of the beam, $B_1 = R_{56}^{(1)} k_1 \sigma_E/E_0$ and $B_2 = R_{56}^{(2)} k_1 \sigma_E/E_0$ for the dimensionless strengths of chicanes, $p = (E - E_0)/\sigma_E$ as the dimensionless energy deviation variable (not to be confused with momentum used in the previous sections) and $\zeta = k_L z$ as a longitudinal coordinate in the beam, with $k_L$ the wave number of the laser (assumed equal in both modulators). The distribution function after the first chicane is given by the following equation (see [2])

$$f_1(\zeta, p) = \frac{1}{\sqrt{2\pi}} e^{-(p-A_1)^2/(2\sigma_p^2)}.  

(19)

Plot of this function for $A_1 = 3$ and $B_1 = 8.47$ (the value of $B_1$ is an optimized value for the 50th harmonic EEHG) is shown in Fig. 1. In order to solve the diffusion equation (15) for $f$ we make the Fourier transformation over the energy variable

$$\hat{f}(\zeta, \mu, s) = \int_{-\infty}^{\infty} dpe^{ip\mu} f(\zeta, p, s).  

(20)

The Fourier transformed Eq. (15)

$$\frac{\partial \hat{f}}{\partial s} = \frac{1}{2\sigma_E^2} D \mu^2 \hat{f}.$$  

(21)

can easily be solved

$$\hat{f}(s, \zeta, \mu) = \hat{f}_1(\zeta, \mu)e^{-D\mu^2/2\sigma_E^2}. 

(22)

Denoting the distribution function after the drift $l$, at the entrance to the second modulator, by $f_2$ we find

$$f_2(\zeta, p) = \int_{-\infty}^{\infty} d\mu e^{-ip\mu} \hat{f}_1(\zeta, \mu)e^{-S\mu^2/2},  

(23)

with $S = Dl/\sigma_E^2$. The second stage of EEHG carries out the following transformation (see [2])

$$p' = p + A_2 \sin(\zeta), \quad \zeta' = \zeta + B_2 p + B_2 A_2 \sin(\zeta).  

(24)

Figure 1: Distribution function $f_1(0, \Delta E/\sigma_E)$ optimized for the 50th harmonic as a function of the normalized energy deviation.
which transforms \( f_2(\zeta, p) \) into a new distribution function \( f_3(\zeta', p') \). Finally, the bunching factor of the \( m \)th harmonic can be computed as a half of the Fourier harmonic of \( f_3(\zeta', p') \) integrated over energy:

\[
b_m = \frac{1}{2} \frac{1}{2\pi} \int_0^{2\pi} d\zeta' \int_{-\infty}^{\infty} dp' e^{im\zeta'} f_3(\zeta', p'). \tag{25}
\]

Due to the symplecticity of the transformation (24) we can replace integration over \( \zeta' \) and \( p' \) in (25) by integration over \( \zeta, p \) variables which also reverts \( f_3 \) to \( f_2 \):

\[
b_m = \frac{1}{2} \frac{1}{2\pi} \int_0^{2\pi} d\zeta \int_{-\infty}^{\infty} dp e^{im(\zeta,p)} f_2(\zeta, p) \tag{26}
\]

Using (23) we can carry out integration over \( p \) and \( \mu \) (the integration over \( p \) results in the delta function \( \delta(\mu - mB_2) \) which makes the integration over \( \mu \) trivial) and express the result in terms of the function \( f_1 \)

\[
b_m = \frac{1}{2} e^{-S_m^2 B_2^2/2} \int_0^{2\pi} d\zeta e^{im(\zeta+B_2A_2 \sin(\zeta))} f_1(\zeta, mB_2). \tag{27}
\]

We see that the bunching factor with Coulomb collisions can be written as a product of the bunching factor without collisions \( b_m^{(0)} \) and an exponential suppression factor \( e^{-S_m^2 B_2^2/2} \):

\[
b_m = b_m^{(0)} e^{-s/L}, \tag{28}
\]

where \( L = 2\sigma_E^2/Dm^2B_2^2 \).

**NUMERICAL EXAMPLES AND CONCLUSION**

For illustration we consider a soft x-ray EEHG FEL scheme with emittance \( \epsilon = 1 \mu m \), beam peak current of 1 kA, and the rms energy spread 100 keV. We also assume the rms transverse bunch size of 100 \( \mu m \). Eq. (18) then gives \( D = 3.1 \text{ keV}^2/	ext{m} \). We considered three EEHG scenarios with the harmonic number \( m = 50, 100 \) and 200. For all 3 cases we assumed that the dimensionless modulation amplitude were \( A_1 = 3 \) and \( A_2 = 6 \). The optimized values of \( B_1 \) and \( B_2 \), the bunching factors without collisions, and the decay distance \( L \) are shown in Table 1. The exponential decay with the three values of \( L \) from Table 1 are shown in Fig. 2. Figure 3 shows smearing out of the distribution for 50th harmonic after the distance \( s = 0.4L \).

As one can see from these results, Coulomb collisions represent a serious limiting factor for the EEHG seeding in the range of harmonic numbers exceeding \( 10^2 \).

**ACKNOWLEDGMENTS**

I would like to thank Z. Huang for useful discussions.

**REFERENCES**