SHORT-RANGE WAKEFIELDS OF SLOWLY TAPERED STRUCTURES*

Boris Podobedov#, Brookhaven National Laboratory, Upton, NY 11973, USA

Abstract

We present new analytical results for short-range geometric wakefields of slowly tapered accelerator structures in 2D geometry.

INTRODUCTION

Wakefields, and in particular geometric wakefields, are important for high intensity beam dynamics in accelerators. Since tapering is routinely used to reduce the wakefields, understanding the taper wakefield is of great value. Wakefields due to tapered structures were studied by many authors (see [1-3] for references and reviews).

Nevertheless, the regime of very short bunches as well as small tapering angles, notoriously difficult to calculate with EM solvers, has not been fully explored. Understanding of this regime becomes more pressing since the bunches are getting shorter while many modern accelerator structures, i.e. superconducting cavities, smoothly tapered collimators, etc., often have the tapering angle gradually varying all the way down to zero.

Another motivation for this paper comes from the recent work [4-5], that showed how to find the wakefields due to arbitrarily short bunches and arbitrary geometry, as long as all wakefield singularities are known. We remind that the wakes near the origin are usually dominated by singularities. For instance, all collimator-like structures obey the so-called optical model, and, in 2D case, the point-charge wakefield is [1]

$$W_{opt}^{\delta}(z) = -Z_0 c\,\pi^{-1}\,\mathrm{Log}(r_{\max}/r_{\min})\,\delta(z)\,, \qquad (1)$$

where Z_0 is the free space impedance, r_{max} and r_{min} are the maximum and minimum radii, and $\delta(z)$ is the delta-function. Similarly cavity-like structures obey the diffraction model [1],

$$W_d^{\delta}(z>0) = -Z_0 c\,\pi^{-2}\sqrt{g/2}\,r_{\min}^{-1}\,z^{-1/2}\,, \qquad (2)$$

where g denotes cavity length. (In our sign convention $W^{\delta}(z)<0$ corresponds to the energy loss of a unit test charge that is trailing distance $z>0$ behind the driving charge.)

As we will show in this paper, the wakefield of smoothly tapered structures (with 1st derivative of the radial boundary matched to zero at the minimum cross-section joints) has an additional singularity, $W^{\delta}(z)\sim z^{-1/3}$. This singularity must be accounted for to find the wakefields due to smooth structures relying on method of [4-5].

In the rest of the paper we sketch the derivation of new analytical results for short-range geometric wakefields of linearly or smoothly tapered structures, comparing them with the results of EM code ECHO [6]. Full version of this work will be published shortly [7].

*Work supported by DOE contract DE-AC02-98CH10886
#boris@bnl.gov

LINEAR TAPERS

Single Linear Taper

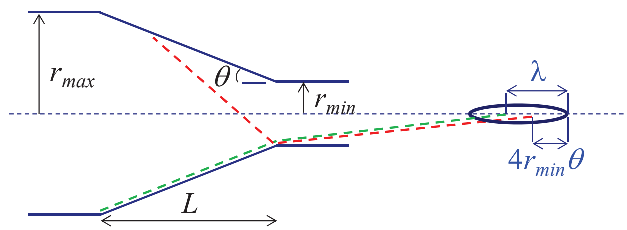

Figure 1: Geometry and length scales in the wake.

We start with an axially symmetric linear taper which transitions from r_{max} to r_{min} over the distance L (Fig.1). The wakefield of this taper, $W^{\delta}(z)$, is fully defined by these three geometric parameters, and they (or their linear combinations with coefficients of order 1) often directly show up as characteristic length scales in $W^{\delta}(z)$.

However, in the important case of a small angle taper, $\theta \ll 1$, much shorter length scales may show up in the short-range wake. Their origin is illustrated in Fig. 1.

The length scales simply follow from causality. Beam fields scattered at the beginning of the taper and then propagating as spherical wave-fronts (i.e. a ray sketched in green) will never catch up with the head of the bunch, because they must "clear the corner" at the very end of the taper. In the process they acquire at least λ/c of delay, where

$$\lambda = \sqrt{(r_{\max}-r_{\min})^2 + L^2} - L \approx \tfrac{1}{2}L\theta^2\,, \qquad (3)$$

so the length scale λ emerges in the front portion of the wakefield. Since, for $z<\lambda$, the waves from the beginning of the structure haven't caught up with the bunch, the wake at these distances cannot explicitly depend on L or r_{max}. Instead, the geometric parameter dependence enters $W^{\delta}(z<\lambda)$ only through r_{min} and θ. If we define "*pre-catchup*" to mean "before the beam fields scattered from the entire length of the transition had a chance to catch-up with the bunch", then λ is simply the length of that bunch. Equivalently, λ is the length of the leading, pre-catchup part of the point-charge wakefield. Similar parameter can be introduced if a tapered transition (linear or not) is a part of a more complicated geometry.

For typical accelerator structures λ is the only length scale associated with the pre-catchup wake. For geometries with large cross-sectional variation, $r_{max}>9r_{min}$, a shorter length scale, $4r_{min}\theta$, shows up as well, because beam fields scattered by the taper, may "cut across" (as shown in Fig. 1 in red) and acquire a minimum delay of $4r_{\min}\theta/c<\lambda/c$. This would produce a peak in the wakefield at $z=4r_{\min}\theta$. For even larger

ISBN 978-3-95450-115-1

r_{max}/r_{min} ratios, multiple (k>1) cuts across along the taper could be possible, resulting in features in the pre-catchup wakefield located at $z=\nu_k$, where

$$\nu_k = (k-1+2^{k-1})4r_{\min}\theta, \quad k>0, \quad \nu_k<\lambda. \quad (4)$$

We emphasize, that for typical structures λ is the only length scale that appears in the pre-catchup wake, while additional, shorter length scales appear only when r_{max}>9r_{min}. By causality, no shorter length scales can be present in this wakefield for z<min(λ, ν_1).

Since for $z<\lambda$ the wakefield depends on only two independent parameters, r_{min} and θ, we expect $W^{\delta}(z<\lambda)$ to have a simple form, which is indeed the case. From dimensional analysis, $W^{\delta}(0+)$ must scale as $Z_0 c/r_{\min}$ multiplied by a unitless factor. As was recently shown in [8], if all longitudinal dimensions of a small-angle tapered structure are scaled by a factor a, the new wakefield is related to the old one by

$$W^{\delta}\left(z;\, a\bar{L}\right) = aW^{\delta}(az;\, \bar{L}). \quad (5)$$

Applying this to a small angle taper, we conclude that

$$W^{\delta}(\lambda \ge z > 0) = \frac{Z_0 c}{2\pi r_{\min}\theta} f\left(\frac{z}{r_{\min}\theta}\right). \quad (6)$$

Additional considerations [7] show that the function $f(u)$ obeys $f(0+)=1$ and $f'(0+)\approx -1$.

In Fig. 2 we compare Eq. (6) with ECHO results for geometry of Fig 1, with fixed θ=0.1 and L=5 cm. To illustrate the convergence to point-charge wakefields, (scaled) wake-potentials for two different bunch lengths are plotted for each geometry. The curves overlap up to $z=\lambda$ and, near the origin, they tend to 1 and -1 slope thus fully confirming Eq. (6).

For r_{max}<9r_{min} (magenta and blue) the pre-catchup wake is especially simple, monotonically decaying up to $z=\lambda$, where it sharply drops further. For larger r_{max}/r_{min} (green and red) a peak shows up at $z=4r_{min}\theta<\lambda$.

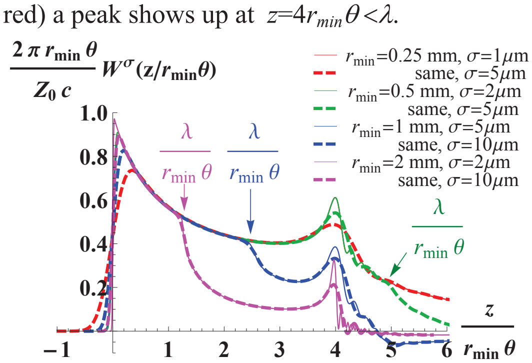

Figure 2: ECHO wake-potentials scaled per Eq. (6).

To our knowledge, Eq. (6) is a new result, describing the wakefield of a taper in a previously unexplored regime. For short bunches, $\sigma << \min(\lambda, 4r_{\min}\theta)$, the energy loss factor due to this wakefield (normalized to that due to the optical model) is

$$k_l / k_{l\_opt} = -\sqrt{\pi}\sigma / \left(2r_{\min}\theta\, \mathrm{Log}\left(r_{\max}/r_{\min}\right)\right). \quad (7)$$

Unlike the optical model wake, the short-range wake of a taper-in is accelerating. For long symmetrically tapered collimator one needs to double the rhs above and add 1.

Eq. (7) differs significantly from previously published expressions (taken to the short bunch limit) derived analytically [2], or by empirical analysis [9].

Adjacent Tapers and Other Structures

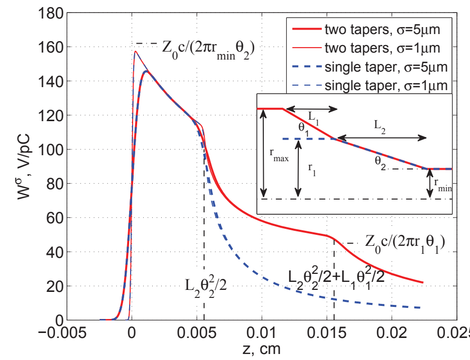

Figure 3: Geometries (inset) and ECHO wake-potentials.

The analysis above can be easily extended to more complicated geometries. For instance, a piece-wise linear transition from $r_{\max} = r_{\min} + L_1\theta_1 + L_2\theta_2$ to $r_{\min}$ shown in Fig. 3 (inset, red), must have the same wake as just the inner taper of this transition (inset, blue dash) up until z reaches the value of $\lambda = \lambda_2 = L_2\theta_2^{\,2}/2$ due to this inner taper. This and further conclusions are illustrated in Fig. 3 by the ECHO results calculated for r_{min}=2 mm, L_1=2 cm, θ_1=1/10, L_2=3.6 cm and θ_2=1/18. Repeating the analysis describing Fig. 1 (and focusing on the convex case, $\theta_1 > \theta_2$) we conclude that the pre-catchup wake of the dual taper lasts up to z equalling the sum of λ parameters for both tapers,

$$\lambda_{dt} = \lambda_1 + \lambda_2 = \tfrac{1}{2}L_1\theta_1^{\,2} + \tfrac{1}{2}L_2\theta_2^{\,2}. \quad (8)$$

The wake at the origin is given by Eq. (6), while $W^{\delta}(z=\lambda_{dt})$ is approximately given by Eq. (6) still taken at z=0 but with r_{min} replaced with the minimum radius of the outer taper. This approximation becomes more and more precise, if all the radii get closer and closer.

The fact that the wakes due to a single and double taper-ins coincide up to $z=\lambda_2$ illustrates that Eq.(6) is applicable to any structure that has linear tapering near the minimum radius, as long as one limits z not to exceed the parameter λ due to this tapering. If the tapering occurs in a (long) symmetric collimator geometry, to get the total wakefield one must double Eq.(6) and add the optical model, Eq. (1). Similarly, for symmetrically tapered cavity-like geometries, one must add the diffraction model, Eq. (2) to Eq. (6). For such structures, Eq. (3) must be replaced with

$$\lambda_{cav} = \tfrac{1}{2}L\theta^2 / (1 - L/g), \quad (9)$$

where L is the length of each taper and g is the cavity length (at r=r_{min}, tapered and un-tapered parts included).

ISBN 978-3-95450-115-1

SMOOTH NON-LINEAR TAPER

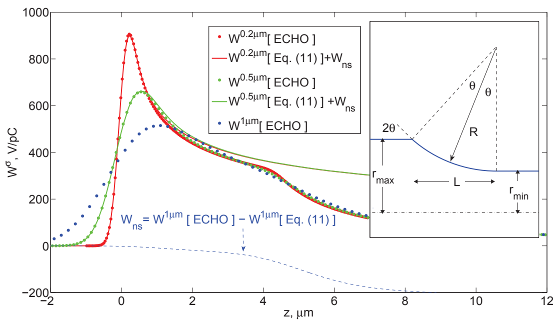

Figure 4: Wake-potentials due to smooth geometry (inset).

Consider non-linear taper with first derivative of the boundary matched to zero at the minimum cross-section (we call this "smooth" tapering). Eq. (6) is not directly applicable here. However, approximating such structures as a sequence of linear tapers, we can extend the analysis above to this case as well.

For instance, consider a circular arc segment of extent 2θ<<1 and radius R (Fig. 4, inset). It can be approximated by n linear tapers of equal length $L/2^{n-1}$. For n=1 it is a linear taper with angle θ and $L \approx 2R\theta$, whose wakefield is given by Eq.(6). For n=2, it is two tapers with $\theta_1 = \frac{3}{2}\theta, \theta_2 = \frac{1}{2}\theta$ of length $L/2$ each. Similar case was illustrated in Fig. 3. Extending Eq. (8) and taking n to infinity we find the pre-catchup wakefield length to be

$$\lambda_s = 2R\theta - L \approx 2L\theta^2/3. \qquad (10)$$

From equations shown in Fig. 3, we see that a fixed z<<λ_s can be approximated by a sum of λ parameters due to the first k<n inner sub-tapers. Thus, for k>>1, $k\sim 2^n z^{1/3}$ and $W^\delta(z)\sim 2^n/k$ leading to $W^\delta(z)\sim z^{-1/3}$. Detailed analysis [7] results in the singular part of the wakefield given by

$$W^\delta(z << \lambda_s) = \frac{Z_0 c}{2\pi r_{\min}}\left(\frac{R}{6z}\right)^{1/3} = \frac{Z_0 c\,(6r''(0-)z)^{-1/3}}{2\pi r_{\min}}, \quad (11)$$

where $r(s)$ is the radial boundary and s=0 is the end of the taper-in. The rhs form of Eq. (11) is applicable to arbitrary boundaries as long as $r'(0-)=0$, $r''(0-)>0$, and higher derivatives could be neglected.

Similar derivation, leading to $W^\delta(z)\sim z^{-m/(2m+1)}$, can be done for structures smooth up to the m^{th} derivative of the boundary, as well as for the infinitely smooth case [7].

The wake-potential due to Eq. (11) can be expressed via hyper-geometric functions. In Fig. 4, it is plotted with ECHO results (for R=20 cm, r_{min} =1 mm, L=1 cm), and the agreement is very good. Since Eq.(11) only includes the singular part of the wakefield, we had to separately find the non-singular part, $W_{ns}(z)$, as explained in Fig. 4.

Eq. (11) (and Eq. (6)) allows a more general scaling than is given by Eq. (5). For instance, if all radial dimensions are scaled by an arbitrary factor b, while all longitudinal ones are scaled by ab^2 (and θ remains small),

$$W^\delta\left(z; ab^2\bar{L}, b\bar{R}\right) = a\,W^\delta(az;\ \bar{L}, \bar{R}). \qquad (12)$$

This scaling for pre-catchup regime wakefield can be shown to hold for all slowly tapered structures, including tapered collimator and cavity-like geometries, since the optical and diffraction model wakes scale proportionally.

Finally, we remark that the short bunch loss factor due to (accelerating) wakefield of Eq. (11) is $\sim r_{min}^{-1}\sigma^{-1/3}$. For small σ and r_{min} the energy gain could be very big.

DISCUSSION

We present new analytical results for short-range geometric wakefields of slowly tapered accelerator structures. We establish the length scale parameter λ that gives the length of the wakefield in the pre-catchup regime. This wakefield is very simple and, for linear tapering, is given by Eq. (6).

For typical accelerator geometries λ is the only length scale of the wakefield in the pre-catchup regime. For large cross-sectional variation, r_{max}>$9r_{min}$, a shorter length, $4r_{min}\theta$, shows up as well. By causality, no shorter length scales can be ever present for z<min(λ, $4r_{min}\theta$).

We established that for structures with boundary tapered smoothly to and/or from r_{min} the wakefield has $z^{-1/3}$ singularity, which is important for finding their point-charge wakefields by method of [4-5]. Also, within approximations made, the wakefield due to a smooth taper-in structure shows an intriguing property of providing strong acceleration and transverse focusing. While other wakes (resistive wall, surface roughness, etc.) as well as practical considerations may limit the observability of this effect, our analysis explains why it shows up in EM simulations, which often approximate smooth boundaries as elliptical arcs, i.e. assuming discontinuous 2nd derivative. One may have to reassess the validity of such boundary approximations for the case of very short bunches, since real accelerator structure surfaces do not have such discontinuities.

While we presented examples due to a taper-in, our results are directly applicable to other structures, with linear, or smooth tapering near r_{min}, as long as we add appropriate asymptotic model (i.e. optical or diffraction).

We focused here on the longitudinal, but the causality arguments similarly apply to the transverse wake. Thus we expect Eqs. (3, 4, 9, 10) to remain the same, while Eq.(6), Eq. (11) and Eq. (12) to acquire factors $\sim\theta\, r_{min}^{-1}$, $\sim z\, r_{min}^{-2}$ and $a^{-1}b^{-2}$ respectively. Extending this analysis to the case of 3D geometries will be performed in the future.

REFERENCES

[1] K. Ng, K. Bane, in *Accelerator Physics and Engineering*, 3rd print, A. Chao, M. Tigner (Eds.), (World Scientific, 1999), p. 229; or in FERMILAB-FN-0901-APC (2010).
[2] S. Heifets, S. Kheifets, Rev. Mod. Phys. 63 (1991) 631.
[3] G. Stupakov, PAC'09, FR2PBI01, p. 4270 (2009).
[4] B. Podobedov, G. Stupakov, PAC'11, WEP179, (2011).
[5] B. Podobedov, G. Stupakov, LER'2011; http:// lowering2011.web.cern.ch/lowering2011
[6] I. Zagorodnov, T. Weiland, PRSTAB 8 (2005) 042001.
[7] B. Podobedov, to be published.
[8] G. Stupakov, K. Bane, I. Zagorodnov, PRSTAB 14 (2011) 014402.
[9] A. Blednykh, S. Krinsky, PRSTAB 13 (2010) 064401.