

# CLASSICAL AND QUANTUM MECHANICAL ANALYSES ON ELECTROMAGNETIC WAVE EMISSIONS IN THE PLANAR CHERENKOV FREE ELECTRON LASER

H. Fares, Y. Kuwamura, M. Yamada, Kanazawa University, Japan.

## Abstract

In the Cherenkov free electron laser (CFEL), the electron interacted with the electromagnetic (EM) wave can be represented as a point particle or as a spatially spreading electron wave in the classical or quantum mechanical framework, respectively. In this paper, we present analytical expressions to describe the stimulated and spontaneous emissions. Also, we show that the results obtained by using the classical treatment are consistent with those obtained in the alternative quantum analysis.

## INTRODUCTION

The CFEL provides a tuneable light source over a very wide range of frequency from microwave to optical region, including the THz spectral range.

In the CFEL, The authors have been showed that the electron can be represented by a spatially spreading wave. Also, it was shown that the coherent length of electron wave  $\ell$  is approximately same as the separating distance between electrons due to the repulsive Coulomb force. This quantum model of electron wave is implied when the EM wavelength is shorter than the spreading length of electron, as in the case of optical emission. Beside the coherent length of electron wave, we introduced another basic factor which is the electron wave relaxation time characterizing the damping phenomena on the time vibration of the electron wave. Analytical expressions of the gain amplification and the coupling coefficient of the spontaneous emission were presented in [1,2].

In this paper, on the basis of a classical analysis, the stimulated gain amplification and the coefficient of the spontaneous emission are formulated to be excited from current sources of the electron beam. We also introduce relaxation phenomena for variations or modulations of electron velocity and density providing a simple parameterization to define the boundary between the so called transit and steady states solutions of the amplification gain, as well as to provide a direct comparison with the quantum model. This classical analysis can be applied for the CFEL operated basically from sub-mm to cm portion of EM spectrum.

## EXCITATION OF THE EM WAVE

The configuration of the CFEL treated in this paper is illustrated in Fig. 1. In Fig. 1, an electron beam is emitted from an electron gun and runs along a surface of a dielectric slab waveguide. The electric field component of  $E_z$  evanesces into the vacuum region to interact with the electron beam. The classical wave equation for the EM wave is written as

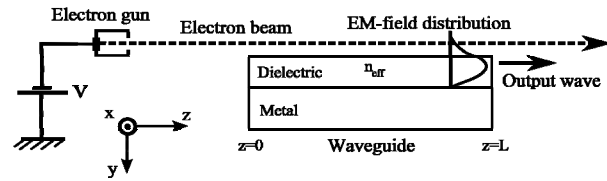


Figure 1: Cerenkov FEL consisting of an electron beam and a slab waveguide.

$$\nabla^2 \mathbf{E} - \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} - \mu_0 \epsilon_i \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial \mathbf{J}}{\partial t}, \quad (1)$$

where  $\epsilon_i$  is the dielectric constant in the  $i$ th layer and  $\sigma$  is the conductivity representing the loss in the waveguide. The interaction with the electron beam is introduced through the current density  $\mathbf{J}$ .

The electric field component  $\mathbf{E}$  of the EM wave is given as

$$\mathbf{E} = F(z) \mathbf{T}(x, y) e^{j(\omega t - \beta z)} + c.c., \quad (2)$$

where  $F(z)$  is the field amplitude of the propagating wave and  $\mathbf{T}(x, y)$  is the transverse electric field distribution.

By substituting Eq. (2) into Eq. (1), taking spatial and time averages after multiplying by  $\mathbf{T}^*(x, y) \exp[j(\beta z - \omega t)]$ , and neglecting the second derivative of  $F(z)$ , we obtain the following equation for the variation of the field amplitude,

$$\frac{\partial F(z)}{\partial z} = j \frac{\mu_0}{2\beta} \frac{1}{\Delta t \Delta z} \times \int_{t-\Delta t}^t \int_{z-\Delta z}^z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial \mathbf{J}}{\partial t} \mathbf{T}^*(x, y) e^{j\beta z} e^{-j\omega t} dx dy dz dt - \frac{\alpha_{\text{loss}}}{2} F(z), \quad (3)$$

where  $\Delta t$  and  $\Delta z$  are time and spatial averages intervals.

In Eq. (3), the current density is divided into two basic components  $\mathbf{J}_{\text{sp}}$  and  $\mathbf{J}_{\text{st}}$  that cause the spontaneous and the stimulated emissions, respectively. So that

$$\mathbf{J} = \mathbf{J}_{\text{sp}} + \mathbf{J}_{\text{st}}. \quad (4)$$

The variation of the propagation power  $P(z)$  is written as,

$$\frac{dP(z)}{dz} = (g - \alpha_{\text{loss}})P(z) + C_{\text{sp}}, \quad (5)$$

where  $g$  is the gain coefficient by the stimulated emission given by

$$g = \text{Im} \left\{ \frac{-\mu_0}{F(z) \beta \Delta t \Delta z} \times \int_{t-\Delta t}^t \int_{z-\Delta z}^z \int_{-\infty}^{\infty} \frac{\partial J_{\text{st}}}{\partial t} T_z^*(x, y) e^{j(\beta z - \omega t)} dx dy dz dt \right\}, \quad (6)$$

and  $C_{\text{sp}}$  is the inclusion rate of the spontaneous emission,

$$C_{\text{sp}} = \sqrt{\frac{\varepsilon_0}{\mu_0}} n_{\text{eff}} \left\{ F(z) \Big|_{\text{sp}} \frac{\partial F^*(z)}{\partial z} \Big|_{\text{sp}} + F^*(z) \Big|_{\text{sp}} \frac{\partial F(z)}{\partial z} \Big|_{\text{sp}} \right\}, \quad (7)$$

$\frac{\partial F(z)}{\partial z} \Big|_{\text{sp}}$  is given by Eq. (3) by replacing  $J$  by  $J_{\text{sp}}$ .

## CFEL IN THE CLASSICAL APPROACH

### The Guided Spontaneous Emission

Here, we denote an electron with a suffix  $i$  and suppose that the electron is spatially localized as a point particle which is represented with a delta-function at a position  $(x_i, y_i, z_i)$  and time  $t_i$ , and that the electron is running with a velocity  $v_i$  along the  $z$  direction. The current density is given by the summation over all electrons as

$$J_{\text{sp}}(r, t) = -e \sum_i v_i \delta(x - x_i) \delta(y - y_i) \delta\{z - v_i(t - t_i)\}, \quad (8)$$

The delta-function can be expanded with an infinite number of spatially harmonic components having spatial variations with propagation constants of  $\beta_1, \dots, \beta_m$ . The field component whose propagation constant is almost the same as  $\beta$  can be built up through the electron motion.

Using Eq. (8) with Eq. (3), the coefficient  $C_{\text{sp}}$  becomes

$$C_{\text{sp}} = \frac{e \bar{J}}{4 \varepsilon_0 n_{\text{eff}}^2} \xi \times \text{Sinc}^2 \left[ (\bar{v} \beta - \omega) \frac{\Delta T}{2} \right]. \quad (9)$$

Here,  $\Delta T$ ,  $\bar{v}$ , and  $\bar{J}$  are the interaction time, the averaged velocity, and averaged current density of the electron beam.  $\xi$  is a coupling coefficient defined as,

$$\xi = \int_{-w}^0 \int_{-w/2}^{w/2} |T_z(x, y)|^2 dx dy \approx w \int_{-w}^0 |T_z(x, y)|^2 dy, \quad (10)$$

From Eq. (9), the broadness of the spontaneous emission profile is determined by the interaction time  $\Delta T$ , and the spectrum has a resonance peak at  $\bar{v} = \omega / \beta$ .

### The Stimulated Emission and Amplification

The stimulated emission is obtained through modulations on the electron velocity and density by the EM field. The variation of the electron velocity  $v$  can be given by

$$\frac{dv}{dt} = -\frac{e}{m_0} \left\{ F(z) T_z(x, y) e^{j(\omega t - \beta z)} + c.c. \right\} - \frac{v}{\tau}, \quad (11)$$

$\tau$  represents the time of the relaxation phenomena whereas the relaxation effect must be caused by the Coulomb repulsing forces among electrons resulting in the electrons-to-electrons scattering [3]. Similarly, the variation of the electron density  $N$  is formulated by

$$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial z} (Nv) - \frac{N - \bar{N}}{\tau}, \quad (12)$$

Here,  $\tau$  works to relax the timely varying component toward the averaged value. Now, supposing that

$$\begin{aligned} v &= \bar{v} + \left\{ u(t) e^{(j\beta \bar{v} - 1/\tau)t - j\beta z} + c.c. \right\} \\ N &= \bar{N} + \left\{ n(t) e^{(j\beta \bar{v} - 1/\tau)t - j\beta z} + c.c. \right\} \end{aligned} \quad (13)$$

where  $\bar{v}$  and  $\bar{N}$  are dc terms of electron velocity and density while  $u(t)$  and  $n(t)$  are amplitudes of the timely varying component. By using Eqs. (11-13) to get the stimulating current density  $J_{\text{st}} = -eNv$ , and using Eq. (3) we can get the time-averaged gain coefficient as

$$\overline{g(\Delta T)} = \frac{\mu_0 \bar{J} e \omega \xi (\Delta T)^2}{m_0} f(\Omega \Delta T), \quad (14)$$

where  $f(\Omega \Delta T)$  is a dispersion function defined by

$$\begin{aligned} f(\Omega \Delta T) &= \text{Im} \left\{ \overline{X(\Delta T)} \right\} / (\Delta T)^2 \\ &= \text{Im} \left\{ \frac{2 \left[ 1 - e^{(j\Omega - 1/\tau)\Delta T} \right] + \left\{ (j\Omega - 1/\tau)\Delta T \left[ 1 + e^{(j\Omega - 1/\tau)\Delta T} \right] \right\}}{(j\Omega - 1/\tau)^3 (\Delta T)^3} \right\}, \end{aligned} \quad (15)$$

$$\Omega = \bar{v} \beta - \omega = \omega (\bar{v} n_{\text{eff}} / c - 1).$$

When the relaxation effect is not taken into account  $\Delta T \ll \tau$ , the dispersion function reduces to

$$f(\Omega \Delta T) \Big|_{\tau \rightarrow \infty} = \frac{2 \{ 1 - \cos(\Omega \Delta T) \} - \Omega \Delta T \sin(\Omega \Delta T)}{(\Omega \Delta T)^3}, \quad (16)$$

which is the well-known dispersion function that shows a peak value of 0.135 at  $\Omega \Delta T = 2.6$ . On the other hand, when  $\Delta T \gg \tau$ , the dispersion function becomes

$$f(\Delta T) \Big|_{\Delta T \gg \tau} = \frac{\tau^2}{(j\Omega \tau - 1)^2} \times \frac{1}{(\Delta T)^2}, \quad (17)$$

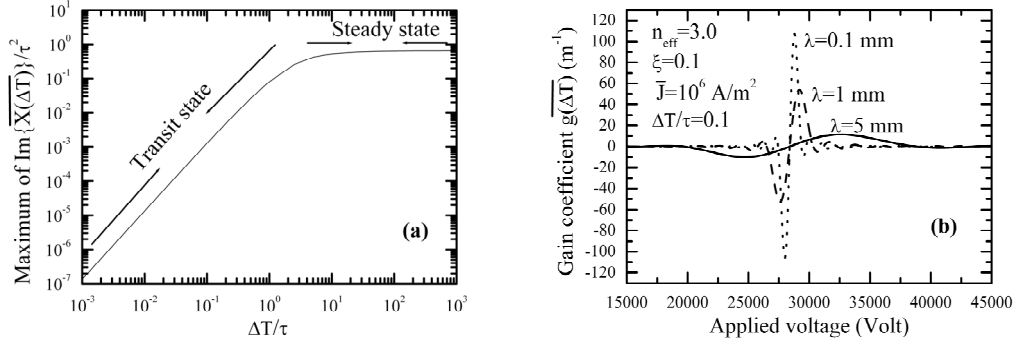


Figure 2: (a). The effect of  $\Delta T$  on maximum values of the normalized dispersion function  $\text{Im}\{\overline{X(\Delta T)}\}/\tau^2$ . (b) The averaged amplification gain coefficient versus the applied voltage at different wavelengths.

The variation of the peak value of  $\overline{X(\Delta T)}$  with  $\Delta T/\tau$  is shown in Fig. 2(a), at which the boundary between the transit state and the steady state is around  $\Delta T/\tau \approx 2$ . Numerical examples of the gain dispersion with the acceleration voltage  $V$  are given in Fig. 2(b), the width of the gain profile is reduced for smaller wavelengths.

## CFEL IN THE QUANTUM APPROACH AND ITS CLASSICAL LIMITS

### The Stimulated Emission Coefficient

In this model the electron has a quantum nature whereas it is represented as a plane wave, for simplicity, the maximum spreading length  $\ell$  of the electron wave is corresponding to the statistical distribution of isolated electrons. The travelling electron wave is written as

$$\Psi_n(t, \mathbf{r}) = (1/\sqrt{\ell^3}) e^{i(\mathbf{k}_n \cdot \mathbf{r} - \omega_n t)}, \quad (18)$$

$\mathbf{k}_n$  and  $\omega_n$  are the wave number and the frequency of the electron wave at an energy level  $n$ . Using the density matrix method to express the dynamic motion of electrons, the gain was estimated in [1] as

$$g = \frac{\mu_0 \hbar}{\beta} \left( \frac{e}{m_0} \right)^2 \frac{\overline{N} \xi_{\text{wave}}}{\tau} \times \sum_b k_b^2 \left[ \frac{\rho_{bb}}{(\omega - \omega_{ba})^2 + 1/\tau^2} - \frac{\rho_{bb}}{(\omega - \omega_{cb})^2 + 1/\tau^2} \right], \quad (19)$$

where  $\rho_{mm}$  is the diagonal element of the density matrix which means the probability to find electrons at the energy level  $m$ .  $\omega_{mn}$  corresponds to the energy difference between levels  $m$  and  $n$ . Note that, the initial, lower, and higher levels are named as  $b$ ,  $a$ , and  $c$ , respectively. Here, the relaxation effect is noticed as the electron wave phase relaxation characterizing the damping phenomenon on the time vibration of the electron wave [4].  $\xi_{\text{wave}}$  is a spatial coupling coefficient between the optical field and single electron wave given by

$$\xi_{\text{wave}} = \int_{-w}^0 \int_{-w/2}^{w/2} \left| \frac{1}{\ell^2} \int_{-\ell/2}^{\ell/2} \int_{-\ell/2}^{\ell/2} T_z(x', y') dx' dy' \right|^2 dx dy, \quad (20)$$

For microwaves when the field distribution is almost constant over  $\ell$ ,  $\xi_{\text{wave}} = \xi$  and that the gain coefficient in Eq. (19) is identical to the gain coefficient in the steady state of the classical model given by Eqs. (14) and (17).

### The Spontaneously Emitted Light Coefficient

The inclusion of the guided spontaneous emission was counted by help of quantization of the optical field to be,

$$C_{\text{spq}} = \frac{e J \tau \Delta \omega}{4 \pi \epsilon_0 n_{\text{eff}}^2} \xi_{\text{wave}} \times \text{Sinc}^2[(k_b - k_a - \beta)\ell/2]. \quad (21)$$

The term  $\Delta \omega$  can be understood as the full width at half-maximum. However, when  $\xi_{\text{wave}} = \xi$ , we find that the quantum mechanical treatment given by Eq. (9) coincides with the result of the classical treatment given by Eq. (21) at the condition of  $\Delta \omega \tau = \pi$  which can be met when  $\tau = \ell/(c/n_{\text{eff}})$ .

## CONCLUSIONS

The classical expressions of the stimulated and spontaneous emissions when the electron is assumed as a point particle are compatible with the results that were already obtained using the quantum model, in which the electron is represented as a plane wave with finite width.

## REFERENCES

- [1] M. Yamada, IEEE J. Quantum Electron. 35(1999) 147.
- [2] Y. Kuwamura, M. Yamada, R. Okamoto, T. Kanai, and H. Fares, J. Appl. Phys. 104 (2008) 103105.
- [3] M. Asada and M. Yamada, J. Appl. Phys. 95 (2004) 5123.
- [4] H. Fares, M. Yamada, Y. Kuwamura, I. Matsumoto, and T. Kanai, IEEE J. of Quantum Electron. 46 (2010) 981.