

# METHOD TO EXTRACT TRANSFER MAPS IN THE PRESENCE OF SPACE CHARGE IN CHARGED PARTICLE BEAMS

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## Abstract

Taylor transfer maps have been standard tools to study accelerator lattices for the last two decades. Taylor transfer maps for static fields, and in the absence of space charge effects, have been derived with different approaches in recent times [1]. However, the derivation of Taylor transfer maps in the presence of space charge is a challenging problem that has no solutions yet. In this paper we present a method for calculating quantities of interest such as tunes, dispersion and resonances from the Taylor map of the system, with an accurate approximation of space charge effects on the beam. The space charge calculation involves a novel numerical integration technique to efficiently evaluate the Taylor expansion of a potential around a singular point. This will allow for an accurate computation of space charge induced tune shifts and resonances, as well as allowing for experimental setups to discriminate between space charge caused issues, and lattice caused issues. The beam optics code COSY Infinity was used to implement the new method. The effects of space charge are added to the map using Strang splitting.

## INTRODUCTION

Space charge is an important effect to consider when designing facilities such as research colliders and free electron lasers; it is so important that machines such as the University of Maryland Electron Ring have been build solely to study space charge. Currently the main methods of simulating space charge involve particle in cell codes which track each macro-particle through the machine only giving information about the test particles we choose to populate the distribution with. When dealing with single particle systems, the machine can be represented with a map ( $\mathcal{M}$ ) which is a way of advancing the system as follows,

$$\vec{z}_f = \mathcal{M}\vec{z}_i, \quad (1)$$

where  $z_i$  and  $z_f$  are the initial and final phase space coordinates. Using this map we can derive, using normal form methods, various quantities of interest. Among other things we can derive the tunes, the chromaticities, the amplitude dependent tune shifts, and the resonance strengths. The purpose of this work is to combine space charge forces with the maps of various beam optical elements, so the effects of space charge can be directly calculated from the map. This combination of single and multi-particle methods will allow a greater understanding of the role that space charge plays in the system.

In this paper we will first examine how the distribution is determined, then how the potential is integrated, until

finally the electric field is calculated and used to implement a kick map. Finally, we will briefly overview the tests and verifications done to the code.

## DISTRIBUTION CALCULATION

Below we describe a procedure to compute the two dimensional density function, as a truncated Taylor series, for a distribution given as a discrete set of points. The moments for the distribution can be calculated by,

$$M_{nm} = \sum_i x_i^n y_i^m. \quad (2)$$

Coefficients of the Taylor expansion of the density function  $\rho(x, y) = \sum_i \sum_j C_{ij} x^i y^j$ , are found by comparing,

$$M_{nm} = \sum_i \sum_j \int_{-x_r}^{x_r} \int_{-y_r}^{y_r} C_{ij} x^{n+i} y^{m+j} dx dy, \quad (3)$$

which can be easily integrated giving a system of linear equations allowing us to solve for  $C_{ij}$ . The system is solved by truncated single value decomposition. The potentials due to each monomial are pre-integrated and stored for faster evaluation at runtime.

## POTENTIAL INTEGRATION

The 2D integral, I, to compute the potential at any point  $(x_0, y_0)$  due to a charge distribution  $\rho(x, y)$  in the domain  $D \in [a, b] \times [c, d]$  is given by,

$$I = \int_c^d \int_a^b \rho(x, y) \ln(\sqrt{(x-x_0)^2 + (y-y_0)^2}) dx dy. \quad (4)$$

If the point  $(x_0, y_0)$  is outside the domain  $D$  the standard Runge-Kutta integration techniques can be used to perform the finite integration. Otherwise the integral I has a singularity at  $(x_0, y_0)$  which must be removed using Duffy's transformation [2] before the integration is performed. Below we describe this procedure for removing a singularity.

The Duffy transformation is done by first viewing the integral as being over a square, the square is subdivided into four as shown in Figure 1.

Integral I in Eq. 4 is now transformed into four integrals which can have their limits of integration rearranged such that the equations become the same indefinite integrals, merely with different bounds. Henceforth we will look at the one integral in the series, remembering that all

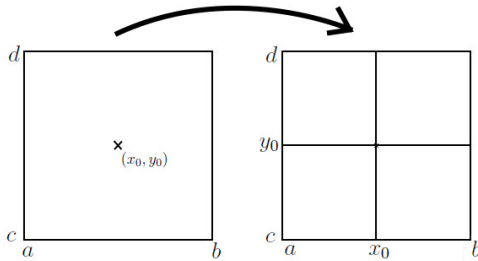


Figure 1: The integration region is subdivided into four smaller regions with their corners on the expansion point.

four will be solved in the same way. We continue the transform by rescaling the square to  $[0, 1]$  on each side, this is done with the coordinate transform,

$$u_1 = \frac{x - x_0}{a - x_0}; \quad u_2 = \frac{y - y_0}{c - y_0}, \quad (5)$$

$$dxdy = (a - x_0)(c - y_0)du_2du_1. \quad (6)$$

For expedience we will use  $\lambda_1 = (a - x_0)$  and  $\lambda_2 = (c - y_0)$ . As shown in Figure 2 we cut the square into triangles which are integrated separately,

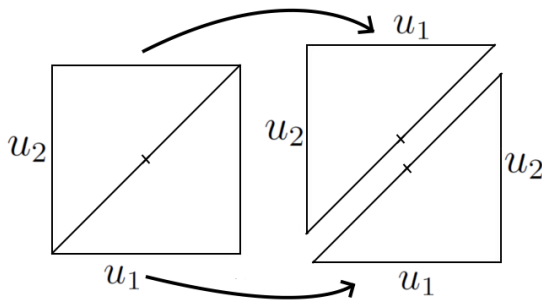


Figure 2: This shows how the integration region is further subdivided into triangles.

$$I_{ac} = \int_0^1 \int_0^{u_1} \lambda_1 \lambda_2 \rho(\lambda_1 u_1 + x_0, \lambda_2 u_2 + y_0) \times \ln(\sqrt{\lambda_1^2 u_1^2 + \lambda_2^2 u_2^2}) du_1 du_2 + \int_0^{u_2} \int_0^1 \lambda_1 \lambda_2 \rho(\lambda_1 u_1 + x_0, \lambda_2 u_2 + y_0) \times \ln(\sqrt{\lambda_1^2 u_1^2 + \lambda_2^2 u_2^2}) du_1 du_2. \quad (7)$$

In order to finish the Duffy transformation the triangles are now converted to unit squares, this is accomplished with the transformations,

$$u_1 = w_1; \quad u_2 = w_1 w_2, \quad (8)$$

and,

$$u_1 = w_1 w_2; \quad u_2 = w_2, \quad (9)$$

which makes the integral take the following form,

$$I_{ac} = \int_0^1 \int_0^1 \lambda_1 \lambda_2 \rho(\lambda_1 w_1 + x_0, \lambda_2 w_1 w_2 + y_0) \times (w_1 \ln(w_1) + w_1 \ln(\sqrt{\lambda_1^2 + \lambda_2^2 w_2^2})) dw_1 dw_2 + \int_0^1 \int_0^1 \lambda_1 \lambda_2 \rho(\lambda_1 w_1 w_2 + x_0, \lambda_2 w_2 + y_0) \times (w_2 \ln(w_2) + w_2 \ln(\sqrt{\lambda_2^2 + \lambda_1^2 w_1^2})) dw_1 dw_2. \quad (11)$$

## ELECTRIC FIELD KICK

Now that the potentials have been calculated the effects must be added to the map. This is done using a method called Strang splitting [3]. This is a way of determining a solution to a differential equation that can be thought of as the sum of two differential equations with known solutions,

$$\frac{d\vec{z}}{ds} = \vec{g}_1(\vec{z}, s) \implies \vec{f}_1(s), \quad (12)$$

$$\frac{d\vec{z}}{ds} = \vec{g}_2(\vec{z}, s) \implies \vec{f}_2(s), \quad (13)$$

which can be solved in the autonomous case as,

$$\frac{d\vec{z}}{ds} = \vec{g}_1(\vec{z}) + \vec{g}_2(\vec{z}) \implies \vec{f}_1\left(\frac{s}{2}\right) \circ \vec{f}_2(s) \circ \vec{f}_1\left(\frac{s}{2}\right) + O(s^3). \quad (14)$$

In the case of space charge the kicks are calculated over a region small enough to allow us to assume an autonomous system. Since the maps for the elements of interest (drifts, quadrupoles, etc.) are already stored in COSY, the element map is used with a kick supplied by the space charge force, or in other cases, other confounding forces such as the earth's magnetic field can be added.

## IN PRACTICE

Now that the fields have been calculated and added to the system, we need to determine how best to use the given parameters to both increase speed and accuracy. One of the main issues is how accurate the simulation is using different values for the moment order and the integration order. As can be seen in Figure 3 there is an ideal area in the high teens for both integration order and moment order for maximum accuracy. The calculated potential was compared to the sum of the Green's functions of each particle.

Since the distribution is being simulated with a series of points, another issue is how many particles are required before the moments converge to their true values. A series of calculations were done which showed that for the lower order terms in a Gaussian distribution the moments converge at approximately the one million particle level.

Another issue that can affect the way that the map elements are calculated is the size of the integration region, care must be taken that the integration region closely

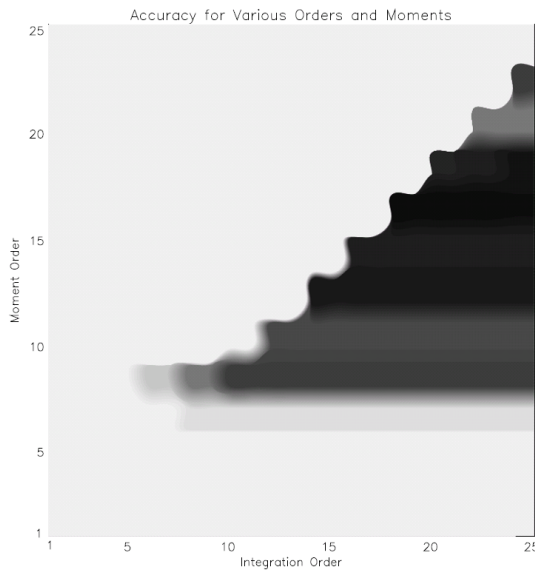


Figure 3: The gradient shows the range covering [0,.003] in accuracy. The darker region denotes higher accuracy.

matches the distribution, since making it too large can modify the values by altering the charge density, and by washing out the intricacies of the distribution. As can be seen in Figure 4 up to the radius of the region the value for the linear horizontal map element is basically constant, but begins to drop off after the boundary is reached.

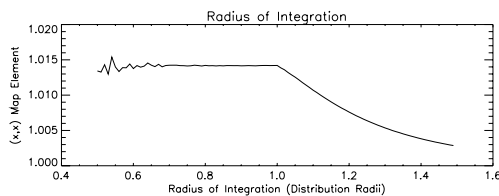


Figure 4: This plot shows how the final position with respect to initial position changes with the size of the integration region.

Now that the radius of integration, the number of particles, and the order of integration and moment calculation has been found, it is time to begin making measurements of the effects that space charge has on the map. In this case we will examine how space charge effects the amplitude dependent tune shift. The beamline in question is a FODO cell with a uniform round distribution; in Figure 5 we see that the amplitude dependent tune shifts are quite small, while in Figure 6 the effect is quite a bit more pronounced due to space charge.

### CONCLUSIONS

As has been shown in this work, putting the effects of space charge into a transfer map, while difficult, is feasible. This method allows for the application of Normal

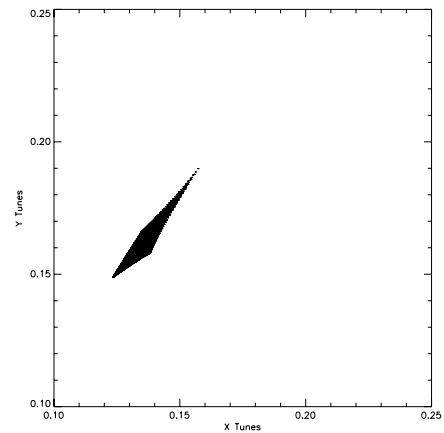


Figure 5: This plot shows a  $2\sigma$  tune footprint of a simple FODO lattice with  $0 \text{ A/M}^2$  current.

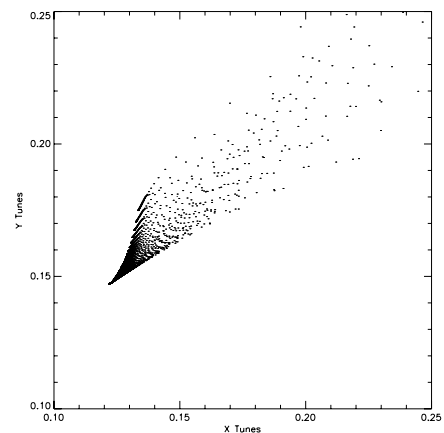


Figure 6: This plot shows a  $2\sigma$  tune footprint of a simple FODO lattice with  $200 \text{ A/M}^2$  current.

Form methods to calculate quantities of interest, including amplitude dependent tune shifts. Further work includes extending this method into three dimensions, and implementation of a fast multipole method to advance larger numbers of particles through a long channel.

### ACKNOWLEDGEMENTS

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