# APPLICABILITY OF PANOFSKY-WENZEL THEOREM

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## Abstract

The Panofsky and Wenzel theorem [1] play very important role in accelerator physics. The well-known conclusion of this theorem is that in a TE mode the deflecting impulse imparted by the electric field always cancels the impulse given by the magnetic fields. In this presentation the Panofsky and Wenzel's formula is elaborated and analyzed obtained correction terms to the transverse kick. As it turned out the net transverse kick for the TE mode is not zero but determined by a ponderomotive force and initial transverse speed spread. Possible consequences of these results are discussed.

## INTRODUCTION

In article [1] it was been derived a relation for the net transverse kick experienced by a fast charge particle crossing a closed cavity excited in a single RF mode

$$\Delta \vec{p}_\perp = e \int_0^L \vec{\nabla}_\perp A_z \Big|_{t=z/v_z} dz \qquad (1)$$

where $e$ is the charge of particle, $z$ is the longitudinal coordinate, $t$ is the time, $\vec{\nabla}_\perp A_z$ is the transverse gradient of $z$-component of RF vector potential $\vec{A}$, $L$ is the length of cavity, $v_z$ is the longitudinal velocity close to the speed of light $c$. Later this relation, usually referred to the Panofsky-Wenzel theorem, was generalized for cavity containing wake field induced by a driving charge [2]. Some reformulated versions of this theorem are given in [3] for study of RF asymmetry in photo-injectors. This theorem plays very important role in accelerator physics. One well-known conclusion followed from Eq.(1) is that in a TE mode ($A_z = 0$) the net transverse kick is zero since the deflecting impulse imparted by the electric field cancels the impulse imparted by the magnetic fields.

However, as it has been shown in [4], if $A_z$ is zero or small enough, the formula (1) may be not true. The fact is that the Panofsky-Wenzel theorem assumes in its derivation that the particle experiencing Lorentz force moves parallel to the z-axis at constant velocity $\vec{v} = \vec{v}_z + \vec{v}_\perp \approx \vec{v}_z$. In this paper we will repeat more exactly the Panofsky-Wenzel's relation, and study conditions, which need to take into account the transverse component of velocity of the particle $\vec{v}_\perp$ during its transit time through the cavity. We will also discuss possible consequences of such consideration

## DERIVING CORRECTION TERMS

Following to Ref. [4], the equation of motion of the particle in terms of a vector potential is given as

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$$\frac{d\vec{p}}{dz} = \frac{e}{v_z} \left[ -\frac{\partial \vec{A}}{\partial t} + \vec{v} \times \vec{\nabla} \times \vec{A} \right] \Bigg|_{t=z/v_z} \qquad (2)$$

where $dz = v_z dt$. Using the following expressions $\vec{v} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{v}\vec{A}) - (\vec{v}\vec{\nabla})\vec{A}$, $\frac{\partial \vec{A}}{\partial t} = \frac{d\vec{A}}{dt} - (\vec{v}\vec{\nabla})\vec{A}$, and expressing the particle velocity as $\vec{v} = \vec{v}_z + v_z \vec{p}_\perp / p_z$, (where $\vec{p}_\perp$ and $p_z$ are the transverse and longitudinal momentums, respectively) we can write the equation for transverse momentum as

$$\frac{d\vec{p}_\perp}{dz} = e \left\{ -\frac{d\vec{A}_\perp}{dz} + \vec{\nabla}_\perp \left( A_z + \frac{\vec{p}_\perp}{p_z} \vec{A}_\perp \right) \right\} \Bigg|_{t=z/v_z} \qquad (3)$$

Here, it should be noted that, as rule, the last term in RHS of Eq.(3) is neglected due to smallness of the absolute value of transverse velocity with respect to longitudinal one ($p_\perp / p_z = v_\perp / v_z \ll 1$). This is justified if the inequality is satisfied

$$\left| \vec{v}_\perp \vec{A}_\perp \right| \ll \left| v_z A_z \right| \qquad (4)$$

However, in the case of the TE mode $A_z = 0$ the last inequality is violated. Therefore, in this case and in more general one, when $v_z A_z \sim \vec{v}_\perp \vec{A}_\perp$, the transverse momentum of the particle should be taken into account in RHS of Eq.(3).

Further, integrating Eq.(3) we obtain the dependence of the transverse momentum on a coordinate $z$

$$\vec{p}_\perp(z) = \vec{p}_{0,\perp} - e\vec{A}_\perp(\vec{r}_\perp, z, t)\Big|_{t=z/v_z}$$
$$+ e\int_0^z \vec{\nabla}_\perp \left( A_z(\vec{r}_\perp, z, t) + \frac{\vec{p}_\perp}{p_z} \vec{A}_\perp(\vec{r}_\perp, z, t) \right) \Bigg|_{t=z/v_z} dz, \qquad (5)$$

where it is assumed that $\vec{A}_\perp = 0$ at $z=0$ and $z=L$ (the cavity end walls are normal the z-diraction or the path of the particle begins and ends in a field-free region), $\vec{p}_{0,\perp}$ is the initial transverse momentum, $\vec{r}_\perp$ is the transverse coordinate of the charge. Due to the small parameter $p_\perp / p_z \ll 1$, the integral equation Eq.(5) may be solved by the successive approximations. Therefore we expand it into series on the small parameter

$$\vec{p}_\perp \approx \vec{p}_{0,\perp} - e\left[1 + \delta\vec{r}_\perp \cdot \vec{\nabla}_\perp\right]\vec{A}_\perp\left(\vec{r}_{0\perp}, z, t\right)\Big|_{t=z/v_z}$$
$$+ e\int_0^z \vec{\nabla}_\perp\left[1 + \delta\vec{r}_\perp \cdot \vec{\nabla}_\perp\right]A_z\left(\vec{r}_{0\perp}, z, t\right)\Big|_{t=z/v_z} dz \qquad (6)$$
$$+ e\int_0^z \vec{\nabla}_\perp \frac{\vec{p}_\perp}{p_z}\vec{A}_\perp\left(\vec{r}_{0\perp}, z, t\right)\Big|_{t=z/v_z} dz.$$

Here it is defined $\delta\vec{r}_\perp$ as $\delta\vec{r}_\perp = \int_0^z \left(\vec{p}_\perp / p_z\right) dz$ and assumed that $\left|\left(\delta\vec{r}_\perp \cdot \vec{\nabla}_\perp\right)\vec{A}\right| \ll \left|\vec{A}\right|$, $\vec{r}_{0,\perp}$ is the initial transverse coordinate of the charge.

Then from Eq.(6) we find the zero order approximation of the transverse momentum as function of $z$-coordinate

$$\vec{p}_\perp^{(0)}\left(z\right) = \vec{p}_{0,\perp} - e\vec{A}_\perp\left(\vec{r}_{0\perp}, z, t\right)\Big|_{t=z/v_z}$$
$$+ e\int_0^z \vec{\nabla}_\perp A_z\left(\vec{r}_{0\perp}, z, t\right)\Big|_{t=z/v_z} dz. \qquad (7)$$

We see that at $z=L$ the zero order approximation Eq.(7) reduces to the Panofsky-Wenzel formula (1). Substituting Eq.(7) into Eq.(6) we can obtain the transverse momentum of the particle with the accuracy of the first order approximation in the form

$$\vec{p}_\perp = \vec{p}_{0,\perp} - e\vec{A}_\perp\left(\vec{r}_\perp, z, t\right)\Big|_{t=z/v_z} + e\int_0^z \vec{\nabla}_\perp A_z\Big|_{t=z/v_z} dz$$
$$+ e^2\int_0^z \vec{\nabla}_\perp\left(\vec{A}_\perp\int_0^z \vec{\nabla}_\perp A_z\Big|_{t=z/v_z} dz\right)\Big|_{t=z/v_z} \frac{dz}{p_z} \qquad (8)$$
$$+ e\int_0^z \vec{\nabla}_\perp\left(\vec{p}_{0,\perp}\vec{A}_\perp - e\left|\vec{A}_\perp\right|^2\right)\Big|_{t=z/v_z} \frac{dz}{p_z},$$

where $\vec{A} \equiv \vec{A}\left(\vec{r}_\perp, z, t\right)\Big|_{t=z/v_z}$.

From the Eq.(8) we see that in the case of exciting a TE mode $\vec{A}_z = 0$ the net transverse kick imparted to the particle, when it leaves cavity, is

$$\Delta\vec{p}_\perp = \int_0^L \vec{\nabla}_\perp\left(\frac{\vec{p}_{0,\perp}}{p_z}e\vec{A}_\perp - \frac{\left|eA_\perp\right|^2}{p_z}\right)\Big|_{t=z/v_z} dz. \qquad (9)$$

As seen from the Eq.(9), even if $\vec{p}_{0,\perp} = 0$ the ponderomotive force, which is square on the transverse component of vector potential, ensures the non-zero transverse momentum imparted to the particle.

## POSSIBLE PHYSICAL EFFECTS

Let us discuss possible consequences of proposed above consideration.

### FEL and Compton Sources

Let us consider the combined vector potential $\vec{A}_\perp = \vec{A}_{w\perp} + \vec{A}_{r\perp}$ of the wiggler or laser pulse, in a case of the Compton source, $\vec{A}_{w\perp}$, and the radiation fields $\vec{A}_{r\perp}$.

Using Eq.(9) and neglecting initial transverse momentum $\vec{p}_{0,\perp} = 0$ we can write the transverse particle speed as

$$\vec{v}_\perp \approx -\frac{e\vec{A}_\perp}{m\gamma} - \frac{1}{\gamma}\int_0^z \vec{\nabla}_\perp\left|\frac{e\vec{A}_\perp}{m}\right|^2 \frac{dz}{\gamma v_z}. \qquad (10).$$

One can see that the ponderomotive force results in transverse drift velocity of beam particles (the second term of Eq.(10)) in direction from the of the maximum-field-density area that can lead to a beam transverse widening.

The consideration of the initial velocity spread $\vec{v}_{0,\perp} \neq 0$ results in the correction term to the electron energy equation in the FEL theory (for example see [5])

$$\frac{d\gamma}{dt} = \frac{1}{2\gamma m^2 c^2}\frac{\partial}{\partial t}\left|e\vec{A}_\perp\right|^2 + \frac{e^2}{\gamma m^2 c^2}\frac{\partial\vec{A}_\perp}{\partial t}\int_0^z \vec{\nabla}_\perp\left(\vec{v}_{0,\perp}\vec{A}_\perp\right)\frac{dz}{v_z} \quad (11)$$

where $\gamma$ is the Lorentz factor. The second new term in Eq.(11) is of ponderomotive type as well as the first one, and can impact on lasing. The corresponding corrections should be taken into account in the 3D wave equation of the self-consistent FEL theory through the transverse electron current density, which can be written as

$$\vec{j}_\perp = e\sum_i \vec{v}_{\perp i}\delta\left[\vec{r} - \vec{r}_i\left(t\right)\right]$$
$$\approx -e^2\sum_i\left[\frac{\vec{A}_\perp}{m\gamma_i}\delta\left[\vec{r} - \vec{r}_i\left(t\right)\right]\right] \qquad (12)$$
$$+ e^2\sum_i \frac{1}{m\gamma_i}\int_0^z \vec{\nabla}_\perp\left(\vec{v}_{0\perp i}\vec{A}_\perp\right)\frac{dz}{v_{z,i}}\delta\left[\vec{r} - \vec{r}_i\left(t\right)\right],$$

where $i$ is the index of a particle. The second terms of Eq.(12) has resonant character as well as the conventional first one, and can impact on lasing.

### RF Asymmetry in Photo-injectors

Excitation of non-resonant axial-asymmetrical modes is source of RF asymmetry in photo-injectors with axial-symmetrical geometry. In generally, these non-resonant modes are the hybrid modes (HEM). However, the components of TM-like modes ($A_z \neq 0$) dominate in the basic volume of the conventional photo-injector cavities but the components of the TE-like ones ($\vec{A}_\perp \neq 0$) is excited in the aperture between RF cavities, and in a beam pipe close to the RF cavity. Therefore, to estimate contribution of the RF asymmetry to transverse

momentum of relativistic particles of a beam, we can apply Eq. (8)

$$\Delta\vec{p}_\perp = e\int_0^L \vec{\nabla}_\perp A_z\Big|_{t=z/v_z} dz + e^2\int_0^L \vec{\nabla}_\perp\left(\vec{A}_\perp\int_0^z \vec{\nabla}_\perp A_z\Big|_{t=z/v_z} dz\right)\Bigg|_{t=z/v_z} \frac{dz}{p_z}$$

$$+\int_0^L \vec{\nabla}_\perp\left(\vec{p}_{0,\perp} e\vec{A}_\perp - \left|e\vec{A}_\perp\right|^2\right)\Bigg|_{t=z/v_z} \frac{dz}{p_z},$$

(13)

where $L$ is the total length of the photo-injectors with the beam pipe where $\vec{A} \neq 0$.

## Conception of TE Mode Deflector

The Eq.(9) shows that the transverse momentum imparted to the particle by a TE mode dependences on the initial transverse momentum. That may point to ability to measure phase volume of a beam by using a TE mode deflector. Let us rewrite Eq.(9) in components as

$$v_{0,x}a_{xx} + v_{0,y}a_{xy} = f_x,$$
$$v_{0,x}a_{yx} + v_{0,y}a_{yy} = f_y,$$

(14)

where

$$a_{xx} = e\int_0^L \frac{\partial}{\partial x}A_x\Big|_{t=z/v_z}\frac{dz}{v_z}, \qquad a_{xy} = e\int_0^L \frac{\partial}{\partial x}A_y\Big|_{t=z/v_z}\frac{dz}{v_z},$$

$$a_{yx} = e\int_0^L \frac{\partial}{\partial y}A_x\Big|_{t=z/v_z}\frac{dz}{v_z}, \qquad a_{yy} = e\int_0^L \frac{\partial}{\partial y}A_y\Big|_{t=z/v_z}\frac{dz}{v_z},$$

$$f_x = e^2\int_0^L \frac{\partial A_\perp^2}{\partial x}\Big|_{t=z/v_z}\frac{dz}{p_z} - \Delta p_x, \quad f_y = e^2\int_0^L \frac{\partial A_\perp^2}{\partial y}\Big|_{t=z/v_z}\frac{dz}{p_z} - \Delta p_y.$$

(15)

For the case $a_{xx} = 0, \qquad a_{yy} = 0$, the solution of the equation set (14) is

$$v_{0,x} = \frac{f_y}{a_{yx}}, \qquad v_{0,y} = \frac{f_x}{a_{xy}}.$$

(16)

For a case of ultrarelativistic particles, ($v_z = c$, $\gamma \mapsto \infty$) Eqs.(16) can be simplified

$$v_{0,x} = -\frac{m_0 c^2\gamma}{e\int_0^L \frac{\partial}{\partial y}A_x\Big|_{\substack{t=z/c\\y=y_0}} dz}\frac{\Delta y}{l}, \quad v_{0,y} = -\frac{m_0 c^2\gamma}{e\int_0^L \frac{\partial}{\partial x}A_y\Big|_{\substack{t=z/c\\x=x_0}} dz}\frac{\Delta x}{l}.$$

(17)

Here the transverse kick ($\Delta p_x$, $\Delta p_y$) is expressed through a beam deflecting from axis ($\Delta x = x - x_0$, $\Delta y = y - y_0$) in a drift tube of length $l$ which is stationed after the cavity, $\Delta p_x = m_0 c\gamma\Delta x/l$, $\Delta p_y = m_0 c\gamma\Delta y/l$, $m_0$ is the rest mass, ($x_0$, $y_0$) and ($x$, $y$) are the transverse coordinates of a particle at the entry of the cavity and the drift tube exit, correspondingly.

More detailed consideration of this method of beam phase volume measurement with using a rectangular cavity as a TE mode deflector is given in Ref. [4].

## Wake Potential

Using the approach developed above we consider wake fields ($\vec{E}, \vec{B}$) in terms of vector and scalar potentials $\vec{A}, \Phi$

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\Phi, \quad \vec{B} = \vec{\nabla}\times\vec{A}, \quad \vec{\nabla}\vec{A} = 0 \quad .$$

(18)

excited by point charge $q$ traversing the cavity at velocity $\vec{v} = \vec{v}_z + \vec{v}_\perp$, $v_z \approx c$. Let a test charge $e$ follows with the same velocity at distance $s$ from the exciting point-charge $q$. The equation for the kick experienced by the test particle in the wake field may be given

$$\frac{d\vec{p}}{dz} = e\left\{-\frac{d\vec{A}}{dz} + \vec{\nabla}\left(A_z - \frac{\Phi}{v_z} + \frac{\vec{p}_\perp}{p_z}\vec{A}_\perp\right)\right\}\Bigg|_{t=(s+z)/v_z}.$$

(19)

Integrating Eq.(19) over (0, $z$), then, expanding $\vec{p}$ into series on the small parameter $p_\perp/p_z \ll 1$, we find the zero order transverse momentum as function of $z$

$$\vec{p}_\perp^{(0)}(z,s) = \vec{p}_{0,\perp} - e\vec{A}_\perp(z,s) + e\int_0^z \vec{\nabla}_\perp\left(A_z - \frac{\Phi}{v_z}\right)\Bigg|_{t=(s+z)/v_z} dz.$$

(20)

Further for simplicity we assume that the path of the particle begins and ends in a field-free region, $\vec{A}(z=0) = \vec{A}(z=L) = 0$. Substituting Eq.(20) into Eq.(19), and taking into account the definition [2], we obtain the wake potential with the correction terms

$$\vec{W}(s) \equiv \frac{v_z\Delta\vec{p}(s)}{eq} = \frac{1}{q}\int_0^L \left[\vec{\nabla}(v_z A_z - \Phi + U)\right]\Big|_{t=(s+z)/v_z} dz,$$

(21)

where $U$ is the wake correction

$$U \equiv v_z\frac{\vec{p}_\perp^{(0)}}{p_z}\vec{A}_\perp\Bigg|_{t=(s+z)/v_z}.$$

(22)

Substituting Eq.(20) into Eq.(22) we obtain the wake potential correction in the form

$$U = v_z\frac{\vec{p}_{0,\perp}\vec{A}_\perp}{p_z} - v_z\frac{e A_\perp^2}{p_z} + \frac{e\vec{A}_\perp}{p_z}\int_0^z \vec{\nabla}_\perp(v_z A_z - \Phi)dz.$$

(23)

As seen from the Eq.(23) two first correction terms to the wake potential are proportional to $\gamma^{-1}$, whereas the modern wake theory [2] gives the correction terms which are proportional to $\gamma^{-2}$.

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