PROPOSAL OF A RELATIONSHIP BETWEEN DYNAMIC APERTURE AND INTENSITY EVOLUTION IN A STORAGE RING

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Abstract

A scaling law for the time-dependence of the dynamic aperture, i.e., the region of phase space where stable motion occurs, was proposed in previous papers, about ten years ago. The use of fundamental theorems of the theory of dynamical systems allowed showing that the dynamic aperture has a logarithmic dependence on time. In this paper this result, proved by mean of numerical simulations, is used as a basis for deriving a scaling law for the intensity evolution in a storage ring. The proposed scaling law is also tested against experimental data showing a remarkable agreement.

INTRODUCTION

In spite of the many efforts (see, e.g., Ref. [1]), a link between the beam lifetime and the value of the dynamic aperture for a circular machine does not seem to be available, yet. So far, the strategies applied aimed at finding the appropriate diffusive model and to derive from this the diffusion constant, and then the law of time-variation of the beam intensity.

Differently from the standard approach, in this paper a proposal is made to use a scaling law for the dynamic aperture as a function of the number of turns N found in the past [3-6] in view of deriving an expression for the time evolution of the beam intensity. Such a model should be valid whenever non-linear effects are driving the particles' motion. A typical example of the time behaviour of the dynamic aperture $D(N)$ is shown in Fig. 1.

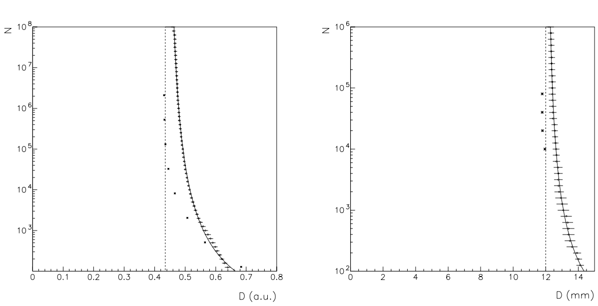

Figure 1: Dependence of the dynamic aperture vs. N for a simple model of nonlinear dynamical system (left, from Ref. [3]) and for a 4D model of the LHC machine (right, from Ref. [5]). The stars represent the prediction of the dynamic aperture by means of Lyapunov exponent.

Assuming a polar grid in normalised phase space

$$x = r\cos\theta \quad y = r\sin\theta \quad \text{with} \quad 0 < \theta < \pi/2, \quad (1)$$

if $r(\theta; N)$ stands for the last stable amplitude up to N turns in the direction θ, then the dynamic aperture reads:

$$D(N) = \frac{2}{\pi}\int_0^{\pi/2} r(\theta; N)\, d\theta \equiv < r(\theta; N) > . \quad (2)$$

According to the results of the studies reported in Refs. [3-6], the following scaling law holds

$$D(N) = D_\infty \left(1 + \frac{b}{[\log N]^\kappa}\right), \quad (3)$$

where D_∞ represents the asymptotic value of the amplitude of the stability domain, while b and κ are additional parameters. These three parameters can be obtained by fitting the results of numerical simulations. It is worth noting that the quantity $D(N)$ is invariant under transformation of the type $N \to N^a$, and $b \to a^\kappa\, b$.

The interesting point is that such a parametrisation is compatible with the hypothesis that the phase space can be partitioned into two regions: a central core, with $r < D_\infty$, where KAM [6] surfaces confine the motion, thus inducing a stable behaviour apart for a set of small measure where Arnold diffusion can take place; an outer part, with $r > D_\infty$, where chaotic motion occurs and the escape rate to infinity is given by a Nekhoroshev-like estimate [7, 8] such as

$$N(r) = N_0 \exp\left(\frac{r_*}{r}\right)^{1/\kappa} \quad (4)$$

where $N(r)$ is the number of turns that are estimated to be stable for particles with initial amplitude smaller than r.

Interestingly enough, two regimes were identified [5]:

- in 4D systems the three quantities D_∞, b, κ are all positive [3, 4]. This corresponds to having a stable region in phase space for arbitrarily long times.
- in 4D systems with tune modulation or off-momentum dynamics it is possible to have no stable region even for a finite number of turns [5]. This corresponds to having the following cases:

$$\begin{cases} D_\infty > 0 & \kappa < 0 & b < 0 \\ D_\infty < 0 & \kappa > 0 & b < 0 \end{cases} \quad (5)$$

SCALING LAW OF BEAM LIFETIME AND LOSSES

The previous picture can be used to derive the variation of the beam intensity due to the particle loss induced by the dynamic aperture. If the beam distribution is assumed to be Gaussian, then by integrating over x' and y' and after changing coordinates (so that the results are expressed in terms of beam sigmas) according to

$$x = \sigma_x \, r \cos\theta \qquad y = \sigma_y \, r \sin\theta \tag{6}$$

and a second integration over θ one obtains the final expression for the beam distribution:

$$\hat{\rho}(r) = r e^{-\frac{r^2}{2}}. \tag{7}$$

By using the very definition of $D(N)$, it is clear that the evolution of the beam intensity $I(N)$ can be found as

$$\frac{I(N)}{I_0} = 1 - \int_{D(N)}^{+\infty} \hat{\rho}(r) \, dr = 1 - e^{-\frac{D^2(N)}{2}}. \tag{8}$$

Positive Dynamic Aperture

This case corresponds to the situation where all the three parameters D_∞, b, κ are positive. Under this assumption, it is readily found that

$$I_\infty = I_0 \left(1 - e^{-\frac{D_\infty^2}{2}} \right). \tag{9}$$

To define the lifetime, it is necessary to convert the expression that rules the variation of the intensity into a pure exponential decay, such as:

$$I(N) = I_0 \, e^{-N/\tau} + I_\infty. \tag{10}$$

By manipulating Eq. (8) it is found

$$\tau = -\frac{N}{\log\left[e^{-\frac{D_\infty^2}{2}} - e^{-\frac{D^2(N)}{2}} \right]}. \tag{11}$$

It is immediately seen that the lifetime τ is indeed a function of N and $\tau(N) \to \infty$ as $N \to \infty$, which is a direct consequence of the fact that a fully stable region exists. Hence, in this scenario the lifetime does not seem to be the best choice of an observable to be linked with the dynamic aperture.

In this respect it is much more relevant to consider simply the expression for the relative losses at time N.

From Eq. (8) it turns out that

$$\frac{\Delta I}{I_0}(N, D_\infty, b, \kappa) = e^{-\frac{D^2(N)}{2}} \tag{12}$$

and the total relative losses are given by

$$\frac{\Delta I}{I_0}(\infty, D_\infty, b, \kappa) = e^{-\frac{D_\infty^2}{2}}. \tag{13}$$

The scaling law for the total relative losses can be easily found. In fact, if $D_\infty \to \alpha D_\infty$, which corresponds to assuming that the dynamic aperture is rescaled or changed due to a change in the dynamical system under consideration, then the losses will scale as

$$\frac{\Delta I}{I_0}(N, \alpha \, D_\infty, b, \kappa) = \left[\frac{\Delta I}{I_0}(N, D_\infty, b, \kappa) \right]^{\alpha^2}. \tag{14}$$

The previous equation shows that the dependence of the losses on the value of the dynamic aperture is rather strong. It is possible to linearise Eq. (14) around the nominal value of the dynamic aperture corresponding to $\alpha = 1$, obtaining

$$\frac{\Delta I}{I_0} \approx \left(\frac{\Delta I}{I_0} \right)_{\alpha=1} + 2 \left(\frac{\Delta I}{I_0} \right)_{\alpha=1} \log \left(\frac{\Delta I}{I_0} \right)_{\alpha=1} (\alpha - 1).$$

Of course, it could be argued that in general, not only the dynamic aperture D_∞ is affected by a change in the system's parameters, but also the constants in the logarithmic law, namely b and κ. In this case, the scaling (14) is exact only for the total losses, as these depend only on the value of D_∞. Indeed, if Eq. (12) is combined with (3), than no simple scaling can be found other than the following

$$\frac{\Delta I}{I_0}(N, D_\infty, b, \kappa) \to \left[I_1 \, I_2^\beta \, I_3^{\beta^2} \right]^{\alpha^2} = \frac{\Delta I}{I_0}(N, \alpha \, D_\infty, \beta \, b, \kappa),$$

with the following definitions

$$I_1 = e^{-\frac{D_\infty^2}{2}} \quad I_2 = e^{-\frac{2 \, b \, D_\infty^2}{2 \, \log^\kappa N}} \quad I_3 = e^{-\frac{b^2 \, D_\infty^2}{2 \, \log^{2\,\kappa} N}}.$$

The invariance property quoted in the introduction implies that the general scaling holds

$$\frac{\Delta I}{I_0}(N^{\beta/\kappa}, \alpha \, D_\infty, \beta \, b, \kappa) = \left[\frac{\Delta I}{I_0}(N, D_\infty, b, \kappa) \right]^{\alpha^2}$$

where the exponent κ is always assumed constant as from a theoretical point of view it should be a function only of the number of degrees of freedom of the system under consideration. Hence, two systems with the same dimensionality can differ only by D_∞ and b.

Zero Dynamic Aperture

This case corresponds to the situation where not all the three parameters D_∞, b, κ are positive. In this situation the relevant quantity is the time at which the dynamic aperture becomes zero, namely

$$D(\bar{N}) = 0 \qquad \log \bar{N} = |b|^{1/\kappa}. \tag{15}$$

From the parametrisation of the inverse logarithm law, $\bar{N}$ is only function of b and κ, but not of D_∞ that is of less importance in such a scenario. From Eq. (15) a scaling law for $\log \bar{N}$ can be derived and it reads

$$\log \bar{N} \to |\beta|^{1/(\gamma \, \kappa)} \, \log^{1/\gamma} \bar{N} \tag{16}$$

if $b \to \beta \, b$ and $\kappa \to \gamma \, \kappa$.

EXPERIMENTAL VERIFICATION

The experimental verification of the proposed scaling law is not so easy as the data available are rather limited. After a search in the literature, an interesting data set was found in Ref. [9]. There, intensity vs. time for the Tevatron with the antiproton beam, only, at injection energy was reported. This data can be analysed using the approach proposed here, as the only source of beam losses is given by the non-linear effects due to the magnets field quality, and hence related to dynamic aperture. The interesting point is that in the paper the curve was analysed assuming a diffusion process those properties are derived in order to match the experimental data. The agreement between experimental data and the proposed model is very good.

As the current proposed approach assumes a pseudo-diffusive behaviour *à la* Nekhoroshev it is natural to believe that a good agreement should be expected, too. Equation (8) is used to fit the data by adjusting the free parameters D_∞, b, and κ. Indeed, in order to make the non-linear fit simpler from the numerical point of view, the various sets of fit parameters D_∞, b where computed for fixed κ, which is then varied and determined by minimising the fit residuals.

The best result is obtained for $D_\infty = 1.1$, $b = 645.6$, and $\kappa = 3.2$, which gives a residue of 4.6×10^{-7}. This should be compared with the residuals for the functions proposed in Ref. [9] that are of the order of $5.5-6.3\times10^{-7}$. This indicates that the proposed approach is at least as good as the standard diffusive models.

In Fig. 2 the experimental data, the fit functions proposed in Ref. [9], and the one based on the inverse logarithm are shown: the agreement is remarkable.

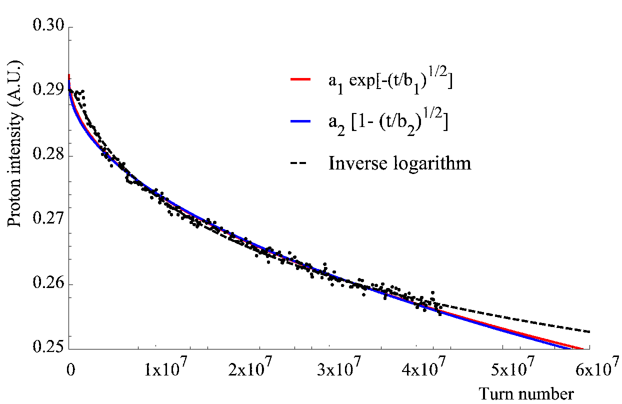

Figure 2: Beam losses at the Tevatron from Ref. [9], interpolated curves proposed therein, and the one proposed in this paper. The agreement is remarkable.

It is worth noting that the asymptotic dynamic aperture is positive, thus indicating that the motion is globally stable. The rather large value of b indicates also that the stochastic motion is occurring in a rather wide region of phase space.

SUMMARY AND CONCLUSIONS

In this paper an attempt to establish a link between the value of the dynamic aperture and the beam losses is presented. Rather than using a diffusive model to describe the particle's motion and hence derive the evolution of the beam intensity, the inverse logarithm decay of the dynamic aperture as a function of turn number is used.

A relationship between the relative losses (total or up to turn number N) and the dynamic aperture is obtained so that a scaling law can be derived for the case of positive dynamic aperture. According to such a scaling law, the dependence on the dynamic aperture is rather strong and a variation of the dynamic aperture around the nominal value induces a large relative variation of the losses.

For the case of zero dynamic aperture, a scaling law of the logarithm of the turn number at which $D(\bar{N}) = 0$ is derived.

The proposed scaling law for intensity vs. time is applied to a data set from Tevatron at injection energy: the agreement is remarkable and at least as good as, if not slightly better than, other standard diffusive models. Of course, it would be nice to collect more experimental data to probe this approach in more details and confirm the encouraging results obtained so far.

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