INVARIANTS OF LINEAR EQUATIONS OF MOTION

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Abstract

Invariants of linear equations of motion generated by second and higher order moments of a beam distribution function are presented in this report.

INTRODUCTION

Courant-Snyder invariant and Root Mean Square (RMS) beam emittance are well-known invariants of linear equation of motion. They are connected with the second order moments of a beam distribution function. Other invariants of linear equations of motion generated by second and higher order moments are presented in this report.

SECOND ORDER INVARIANTS

Considering 2D problem let us introduce the vector $Y^T = (x_1, x_1', x_2, x_2') = (Y_1^T, Y_2^T)$, where superscript T defines transpose vector or matrix, prime denotes derivative with respect to distance s along the beam trajectory. In the linear approximation the components of vector Y satisfies to matrix equations:

$$Y_{1,2}' = A_{1,2} Y_{1,2} \quad ; \quad A_{1,2} = \begin{pmatrix} 0 & E_1 \\ b_{1,2}(s) & 0 \end{pmatrix} \qquad (1)$$

Here E_n (n=1) is unit matrix of n-th order, $b_{1,2}$ are square matrix of n-th order defined by electromagnetic fields [1]. It should be noted that for motion in longitudinal magnetic field representation of the matrices $A_{1,2}$ in form (1) is valid in coordinate frame rotating with Larmor's frequency around the longitudinal axis.

The second order moments M of the beam distribution function f are defined in accordance with formula:

$$M = \overline{YY^T} = \frac{1}{N} \int YY^T f \, dy \qquad (2)$$

Here N is number of particle per unit beam length, integration in (2) is fulfilled over all phase space occupied by particles. In accordance with system (1) matrix M satisfy the equation [1]:

$$M' = AM + MA^T \quad ; \quad A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \qquad (3)$$

The well-known invariants of the system (3) are RMS-emittances $\varepsilon_{1,2}$ [2,3] for both transverse degrees of freedom:

$$\varepsilon_k^2 = \overline{x_k^2}\,\overline{x_k'^2} - (\overline{x_k x_k'})^2 = const \quad ; \quad k = 1,2 \qquad (4)$$

The RMS-emittances (4) are the determinants of matrices $\overline{Y_k Y_k^T}$. It may be shown that the determinant Δ_{12} of matrix $\overline{Y_1 Y_2^T}$ is also constant along the beam trajectory [4]:

$$\Delta_{12} = \overline{x_1 x_2}\,\overline{x_1' x_2'} - \overline{x_1 x_2'}\,\overline{x_1' x_2} = const \qquad (5)$$

Each vector Y_k defines the invariant I_k:

$$I_k = Y_k^T (\overline{Y_k Y_k^T})^{-1} Y_k = const \qquad (6)$$

Here superscript "-1" denotes inverse matrix. The expression (6) is analog of Courant-Snyder invariant. Indeed, by introducing Twiss's parameters according to formula:

$$\overline{Y_k Y_k^T} = \varepsilon_k \begin{pmatrix} \beta_k & -\alpha_k \\ -\alpha_k & \gamma_k \end{pmatrix} \quad , \qquad (7)$$

it can be reduced to standard form:

$$\beta_k x_k'^2 + 2\alpha_k x_k x_k' + \gamma_k x_k^2 = const \qquad (8)$$

The pair of vectors (Y_1, Y_2) produce the "coupling" invariant:

$$I_{12} = Y_1^T (\overline{Y_1 Y_1^T})^{-1}\, \overline{Y_1 Y_2^T}\, (\overline{Y_2 Y_2^T})^{-1} Y_2 = const \quad , \qquad (9)$$

which coincides with (6) in the case $Y_2 = Y_1$.

HIGHER ORDER INVARIANTS

Higher moments of the distribution function $M^{(n)}_{i_n i_{n-1} \cdots i_1}$, where indices i_k vary from 1 to N_p, are introduced according to the definition (2):

$$M^{(n)}_{i_n i_{n-1} \cdots i_1} = \overline{y_{i_1} y_{i_2} \cdots y_n} \quad ; \quad N_p \ge i_1, \cdots, i_n \ge 1 \, , \qquad (10)$$

Here N_p – is the phase space dimension. In accordance with the formula (10) the total number of moments of

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order n is N_p^n. Not all of them are independent, since the product $Z^{(n)}_{i_1 i_2 \cdots i_n}$:

$$Z^{(n)}_{i_1 i_2 \cdots i_n} = y_{i_1} y_{i_2} \cdots y_{i_n} \quad , \tag{11}$$

is symmetric with respect to any permutation of indices i_k.

To resolve this uncertainty, only the independent products $Z^{(n)}_{i_1 i_2 \cdots i_n}$ of order n are taken into account. This means that the product $Z^{(n)}_{i_1 i_2 \cdots i_n}$ can not be obtained from the other by any permutation of indices. To fulfil these conditions indices i_k must satisfy the system of inequalities:

$$N_p \geq i_n \geq i_{n-1} \geq \cdots \geq i_2 \geq i_1 \geq 1 \tag{12}$$

Each product $Z^{(n)}_{i_1 i_2 \cdots i_n}$ of order n with the sequence of indices satisfying the inequalities (12) can be put in one-to-one correspondence with the number i:

$$i = i_1 + \sum_{k=2}^{n} \binom{i_k + k - 2}{k} \tag{13}$$

where $\binom{m}{k} = \dfrac{m!}{k!(m-k)!}$ – are binomial coefficients.

The number N_n of independent products of order n and, therefore, of the moments $M^{(n)}_{i_n i_{n-1} \cdots i_1}$ of order n in accordance with the formula (13) is:

$$N_n = N_p + \sum_{k=2}^{n} \binom{N_p + k - 2}{k} = \binom{N_p + n - 1}{n} \tag{14}$$

The last formula may be proved by induction. The number N_n of moments of order n for the different dimensions of the phase space N_p is given in Table 1.

Table 1: Number of moments of order n

n	$N_p = 2$	$N_p = 4$	$N_p = 6$
1	2	4	6
2	3	10	21
3	4	20	126
4	5	35	252
5	6	56	462
6	7	84	792

The dependence of components of tensor $Z^{(n)}_{i_1 i_2 \cdots i_n}$ on distance s can also be studied by means of matrix formalism. Let us introduce the vector Y_n of dimension N_n:

$$Y_n = \begin{pmatrix} y_1^n \\ y_1^{n-1} y_2 \\ \vdots \\ y_{N_p}^n \end{pmatrix} \quad , \tag{15}$$

which components are independent products (11,12). In the case of linear equation of motion vector Y_n satisfies the system of equations:

$$Y_n' = A_n Y_n \quad , \tag{16}$$

where elements of $(N_n \times N_n)$ matrix A_n are linear combination of elements of matrix A (3). For example, when $N_p = 2$ matrix A_n has the following form:

$$A_n = \begin{pmatrix} 0 & n & 0 & \cdots & 0 \\ -k^2(s) & 0 & n-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & -(n-1)k^2(s) & 0 & 1 \\ 0 & \cdots & 0 & -nk^2(s) & 0 \end{pmatrix} \tag{17}$$

In this case A_1 coincides with (1).

The moments $M^{(n)}_{i_n i_{n-1} \cdots i_1}$ of order n are equal to:

$$M^{(n)}_{i_n i_{n-1} \cdots i_1} = \overline{Y_n} \tag{18}$$

and the equations for them coincide with the system (16).

Each pair of vectors Y_n, Y_m define invariant I_{nm}:

$$I_{nm} = Y_n^T \left(\overline{Y_n Y_n^T} \right)^{-1} \overline{Y_n Y_m} \left(\overline{Y_m Y_m^T} \right)^{-1} Y_m = const \tag{19}$$

Indeed, the moments of (n+m) order $\overline{Y_n Y_m}$ in accordance with the system (7) satisfies the following equations:

$$(\overline{Y_n Y_m^T})' = A_n \overline{Y_n Y_m^T} + \overline{Y_n Y_m^T} A_m^T \tag{20}$$

Using equations (16,20) is easy to show compliance of equality:

$$I_{nm}' = 0 \quad , \tag{21}$$

which implies the invariance of I_{nm}. For n=m=1 formula (19) coincides with the Courant-Snyder invariant (6).

In the absence of damping in addition to the invariants (19) the values of the determinants Δ_n of matrices $\overline{Y_n Y_m^T}$ are integrals of motion. The value of Δ_n varies along s according to the equation:

$$\frac{1}{\Delta_n}\Delta_n' = 2Tr(A_n) \quad , \tag{22}$$

where $Tr(A_n)$ – is trace of matrix A_n. In the absence of damping ($Tr(A_n) = 0$) one can get:

$$\Delta_n = const \tag{23}$$

For n = 1 last formula defines conservation of beam RMS emittance.

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