EXPLICIT MAPS FOR THE FRINGE FIELD OF A QUADRUPOLE*

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Abstract

A perturbation method based on Lie technique, originated by J. Irwin and C.-x. Wang, was extended to calculate the linear maps for the fringe field of a quadrupole. In our method, the fringe field shape is not necessarily anti-symmetric with respect to the hard-edge position. The linear maps were explicitly expressed as functions of fringe field integrals. Thus they can be used to assess the influence of the quadrupole fringe fields in beam dynamics.

INTRODUCTION

Fringe field effects are often neglected when people carry out optics design and numerical simulations for large storage rings, where the fringe field extensions are much smaller than the magnet lengths. For these studies, the conventional hard-edge model [1] is usually chosen. The design of modern light sources and colliders require more accurate modelling of the rings for purpose of studying the long-term beam dynamics accurately. With the fully developed analytical techniques such as Lie algebra [2] and differential algebra [3], it is convenient to calculate the linear and nonlinear maps for s-dependent magnetic fields such as quadrupole fringe fields [4-5].

Irwin and Wang proposed a Lie method to analytically estimate the maps for the fringe field of a quadrupole [4]. Their idea is to treat the fringe field as perturbation on the ideal hard-edge lens and then apply Lie technique in calculating the perturbation maps for fringe field. In their work, the fringe field is assumed to be anti-symmetric with respect to the edge of a normal hard-edge model. But we found that this assumption is not necessary. By re-defining the fringe field integrals, more accurate maps for arbitrary shape of fringe field can be calculated using the same method.

S-DEPENDENT HAMILTONIAN AND FRINGE FIELD INTEGRALS

We start from defining the s-dependent Hamiltonian for the on-momentum particle in a quadrupole as

$$H(s) = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}K(s)\left(x^2 - y^2\right) \qquad (1)$$

where p_x, p_y are the normalized particle momenta and $K(s)$ represents the s-dependent quadrupole strength. Following Irwin and Wang [4], we separate the fringe field region from the body part, i.e.

$$H(s) = H_0(s) + \tilde{H}(s) \qquad (2)$$

where $H_0(s)$ and $\tilde{H}(s)$ are the Hamiltonians of the body described by a hard-edge model and the perturbation due to the fringe field. For the exit-side fringe field, we have

$$H_0(s) = \begin{cases} \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}K_0\left(x^2 - y^2\right), & s_1 \le s \le s_0 \\ \frac{1}{2}\left(p_x^2 + p_y^2\right), & s_0 < s \le s_2 \end{cases} \qquad (3)$$

$$\begin{aligned} \tilde{H}(s) &= \frac{1}{2}\tilde{K}(s)\left(x^2 - y^2\right) \\ &= \begin{cases} \frac{1}{2}\left[K(s) - K_0\right]\left(x^2 - y^2\right), & s_1 \le s \le s_0 \\ \frac{1}{2}K(s)\left(x^2 - y^2\right), & s_0 < s \le s_2 \end{cases} \end{aligned} \qquad (4)$$

where $s_1 = 0$ is the centre of the quadrupole, s_2 is a point far outside the fringe field region, and s_0 is the magnet end point defined by the hard-edge model (see Figure 1).

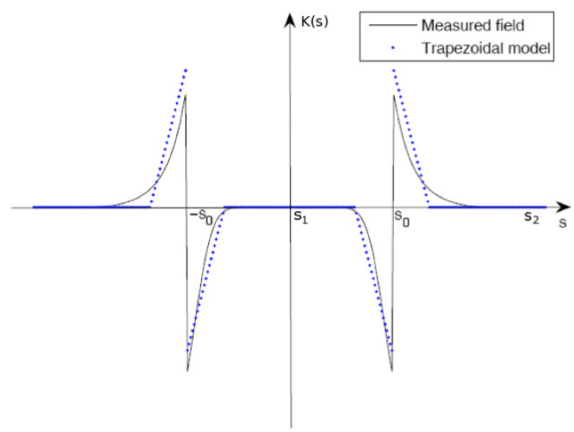

Figure 1: Typical fringe field distribution from field measurements and corresponding trapezoidal model.

The linear map from s_1 to s_2 can be written as

$$M(s_1 \to s_2) = R_-(s_1 \to s_0) R_+(s_0 \to s_2) \qquad (5)$$

$$R_-(s_1 \to s_0) = M_Q(s_1 \to s_0) e^{:f_2^-:} \qquad (6)$$

$$R_+(s_0 \to s_2) = e^{:f_2^+:} M_{drift}(s_0 \to s_2) \qquad (7)$$

where M_Q and M_{drift} are exact linear maps, i.e.

$$M_Q(s_1 \to s) \leftrightarrow \begin{bmatrix} \cos\left(\sqrt{K_0}s\right) & \frac{\sin\left(\sqrt{K_0}s\right)}{\sqrt{K_0}} & 0 & 0 \\ -\sqrt{K_0}\sin\left(\sqrt{K_0}s\right) & \cos\left(\sqrt{K_0}s\right) & 0 & 0 \\ 0 & 0 & \cosh\left(\sqrt{K_0}s\right) & \frac{\sinh\left(\sqrt{K_0}s\right)}{\sqrt{K_0}} \\ 0 & 0 & \sqrt{K_0}\sinh\left(\sqrt{K_0}s\right) & \cosh\left(\sqrt{K_0}s\right) \end{bmatrix} \qquad (8)$$

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$$M_{drift}(s_0 \to s) \leftrightarrow \begin{bmatrix} 1 & s-s_0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s-s_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

Assuming that the fringe perturbation is weak, the generating functions of the linear fringe maps can be calculated using the second-order BCH formulae

$$f_2^- = -\int_{s_1}^{s_0} \bar{H}(s)ds + \frac{1}{2}\int_{s_1}^{s_0} ds \int_{s}^{s_0} ds' \left[\bar{H}(s), \bar{H}(s')\right] \tag{10}$$

$$f_2^+ = -\int_{s_0}^{s_2} \bar{H}(s)ds + \frac{1}{2}\int_{s_0}^{s_2} ds \int_{s}^{s_2} ds' \left[\bar{H}(s), \bar{H}(s')\right] \tag{11}$$

where

$$\bar{H}(s) = \begin{cases} \tilde{H}\left(s, M_Q(s_0 \to s)X\right), & s_1 \le s \le s_0 \\ \tilde{H}\left(s, M_{drift}(s_0 \to s)X\right), & s_0 < s \le s_2 \end{cases} \tag{12}$$

$$X = \left[x, p_x, y, p_y\right]^T \tag{13}$$

where X is the phase space vector at s_0. To calculate f_2^- and f_2^+ explicitly, Taylor expansion around s_0 to the third order is performed to the linear transfer matrix Eq. (8) and we obtain

$$M_Q(s_0 \to s) \leftrightarrow \begin{bmatrix} 1-\frac{1}{2}K_0(s)\Delta s^2 & \Delta s - \frac{1}{6}K_0(s)\Delta s^3 & 0 & 0 \\ -K_0\left[\Delta s - \frac{1}{6}K_0(s)\Delta s^3\right] & 1-\frac{1}{2}K_0(s)\Delta s^2 & 0 & 0 \\ 0 & 0 & 1+\frac{1}{2}K_0(s)\Delta s^2 & \Delta s + \frac{1}{6}K_0(s)\Delta s^3 \\ 0 & 0 & K_0\left[\Delta s - \frac{1}{6}K_0(s)\Delta s^3\right] & 1+\frac{1}{2}K_0(s)\Delta s^2 \end{bmatrix} \tag{14}$$

where $\Delta s = s - s_0$. Again we have assumed here that the fringe field region is short. Then we can calculate the generating functions as

$$\begin{aligned} f_2^- \cong & -\frac{1}{2}I_0^-\left(x^2-y^2\right) - I_1^-\left(xp_x - yp_y\right) \\ & -\frac{1}{2}I_2^-\left(p_x^2 - p_y^2\right) + \frac{1}{2}K_0 I_2^-\left(x^2+y^2\right) \\ & +\frac{2}{3}K_0 I_3^-\left(xp_x - yp_y\right) + \frac{1}{2}\Lambda_2^-\left(x^2+y^2\right) \end{aligned} \tag{15}$$

$$\begin{aligned} f_2^+ \cong & -\frac{1}{2}I_0^+\left(x^2-y^2\right) - I_1^+\left(xp_x - yp_y\right) \\ & -\frac{1}{2}I_2^+\left(p_x^2 - p_y^2\right) + \frac{1}{2}\Lambda_2^+\left(x^2+y^2\right) \end{aligned} \tag{16}$$

From Eqs. (5-7), we also obtain the total linear fringe map

$$R_f = e^{:f_2^-:}e^{:f_2^+:} = e^{:f_2:} \tag{17}$$

where

$$\begin{aligned} f_2 \cong & f_2^- + f_2^+ + \frac{1}{2}\left[f_2^-, f_2^+\right] \\ \approx & -\left(I_1^- + I_1^+\right)\left(xp_x - yp_y\right) - \frac{I_2^- + I_2^+}{2}\left(p_x^2 - p_y^2\right) \\ & + \frac{K_0 I_2^-}{2}\left(x^2+y^2\right) + \frac{2K_0 I_3^-}{3}\left(xp_x + yp_y\right) + \\ & \frac{\Lambda_2^- + \Lambda_2^+}{2}\left(x^2+y^2\right) - \frac{1}{2}I_0^+\left(I_1^- + I_1^+\right)\left(x^2+y^2\right) \\ & -\frac{1}{2}I_0^+\left(I_2^- + I_2^+\right)\left(xp_x + yp_y\right) + \dots \end{aligned} \tag{18}$$

where we define the fringe field integrals from the zero-order to the third-order as

$$\begin{aligned} I_0^- &= \int_{s_1}^{s_0} \tilde{K}(s)ds, & I_1^- &= \int_{s_1}^{s_0} \tilde{K}(s)(s-s_0)ds \\ I_2^- &= \int_{s_1}^{s_0} \tilde{K}(s)(s-s_0)^2 ds, & I_3^- &= \int_{s_1}^{s_0} \tilde{K}(s)(s-s_0)^3 ds \end{aligned} \tag{19}$$

$$\begin{aligned} I_0^+ &= \int_{s_0}^{s_2} \tilde{K}(s)ds, & I_1^+ &= \int_{s_0}^{s_2} \tilde{K}(s)(s-s_0)ds \\ I_2^+ &= \int_{s_0}^{s_2} \tilde{K}(s)(s-s_0)^2 ds, & I_3^+ &= \int_{s_0}^{s_2} \tilde{K}(s)(s-s_0)^3 ds \end{aligned} \tag{20}$$

$$\begin{aligned} \Lambda_2^- &= \int_{s_1}^{s_0} ds \int_{s}^{s_0} ds' \tilde{K}(s)\tilde{K}(s')(s'-s) \\ \Lambda_2^+ &= \int_{s_0}^{s_2} ds \int_{s}^{s_2} ds' \tilde{K}(s)\tilde{K}(s')(s'-s) \end{aligned} \tag{21}$$

From Eqs. (19-21), we can find that, only when the fringe field is anti-symmetric with respect to s_0, there exist $I_0^- = -I_0^+$, $I_1^- = I_1^+$, and $I_2^- = -I_2^+$. And then the terms of $p_x^2 - p_y^2$ and $xp_x + yp_y$ will cancel out in Eq. (18). But in reality, the fringe field is not anti-symmetric. In this case, the fringe map generated by Eq. (18) should be more accurate.

LINEAR FRINGE MAP

From Eq. (18), we can find the approximated perturbation matrix in the focusing plane due to the exit fringe field

$$M_{r,x} = \begin{bmatrix} 1 & 0 \\ J_3 & 1 \end{bmatrix}\begin{bmatrix} 1 & J_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^{J_1} & 0 \\ 0 & e^{-J_1} \end{bmatrix} \tag{22}$$

where we define

$$J_1 = \left(I_1^- + I_1^+\right) - \frac{2K_0 I_3^-}{3} + \frac{1}{2}I_0^+\left(I_2^- + I_2^+\right) \tag{23}$$

$$J_2 = I_2^- + I_2^+ \tag{24}$$

$$J_3 = K_0 I_2^- + \left(\Lambda_2^- + \Lambda_2^+\right) - I_0^+\left(I_1^- + I_1^+\right) \tag{25}$$

Here we artificially separate the fringe map into three parts: The right-most matrix on the right side of Eq. (22), corresponding to the terms of $xp_x \pm yp_y$ in Eq. (18), is the effect of image magnification elements. This is the leading term of the linear fringe map. The terms $x^2 + y^2$

are the effects of focusing elements, which correspond to the left-most matrix on the right side of Eq. (22). The terms $p_x^2 - p_y^2$ are the effects of drift elements due to non-symmetric shape of fringe field, corresponding to the middle matrix on the right side in Eq. (22). The matrix form of Eq. (22) is similar to the symplectic treatment in SAD code [6] on the quadrupole fringe field. However, the focusing effect by J_3 is neglected in SAD.

It is straightforward to perform the same calculation for the entrance-side of fringe field. If $K(s)$ is symmetric with respect to the centre of the quadrupole, we can easily find the perturbation matrix as following

$$M_{l,x} = \begin{bmatrix} e^{-J_1} & 0 \\ 0 & e^{J_1} \end{bmatrix} \begin{bmatrix} 1 & J_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ J_3 & 1 \end{bmatrix} \quad (26)$$

where we used the same definitions as in Eqs. (23-25).

It's trivial to calculate the perturbation matrices in the defocusing plane.

APPLICATIONS OF LINEAR FRINGE MAP

The first application of the fringe maps we have tried is to calculate the effective focal length of a normal quadrupole and the tune shifts induced by quadrupole fringe fields in the focusing plane. Putting the perturbation matrices to both sides of the exact matrix of a hard-edge model, we can obtain the whole focal length of a normal quadrupole as

$$f^{-1} \cong \sqrt{K_0} \sin\left(\sqrt{K_0} L_0\right) e^{-2J_1} - 2J_3 \cos\left(\sqrt{K_0} L_0\right) \quad (27)$$

Obviously this formula is much simpler than the Eq. (27) in Ref. [7]. The correction on the focal length due to the fringe fields is mainly contributed by the image magnification terms.

The second application is to estimate the tune shift due to the fringe fields in a quadrupole. The perturbed one-turn matrix at the exit-side of a quadrupole in a ring can be

$$\begin{bmatrix} \cos(2\pi Q_0) + \alpha \sin(2\pi Q_0) & \beta \sin(2\pi Q_0) \\ -\dfrac{1+\alpha^2}{\beta} \sin(2\pi Q_0) & \cos(2\pi Q_0) - \alpha \sin(2\pi Q_0) \end{bmatrix}_r M_{r,x} = \begin{bmatrix} \cos(2\pi Q) + \alpha \sin(2\pi Q) & \beta \sin(2\pi Q) \\ -\dfrac{1+\alpha^2}{\beta} \sin(2\pi Q) & \cos(2\pi Q) - \alpha \sin(2\pi Q) \end{bmatrix}_r \quad (28)$$

where Q_0 is the non-perturbed tune and Q is the tune with the fringe field perturbation. From Eq. (28), the tune shift due to the exit-side fringe field can be calculated

$$\Delta Q_r = Q - Q_0 \cong -\frac{\alpha_r}{2\pi} J_1 + \frac{1}{4\pi} \beta_r J_3 - \frac{1+\alpha_r^2}{4\pi \beta_r} J_2 \quad (29)$$

where α_r, β_r are the Twiss parameters at the exit-side of the quadrupole. Comparing with the term of J_1, the terms containing J_2 and J_3 in Eq. (29) are higher-order terms and can be neglected in most cases. Similar result as Eq. (29) can be easily obtained for the tune shift due to the entrance-side fringe field.

Again we can repeat the same calculation for the defocusing plane. We can also use the linear fringe maps to calculate the effective length and strength of a quadrupole. This will be discussed in details elsewhere.

DISCUSSION AND SUMMARY

The linear fringe maps in a normal quadrupole were re-calculated by using the Lie technique. The most important idea is to re-define the fringe field integrals, taking into account the non-symmetric profile of the field distribution. Thus high-order terms can be evaluated correctly in the fringe maps. As straightforward applications, simple formulae were found to estimate the tune shifts induced by the fringe fields in quadrupoles. This should be valuable in practical design of the storage ring optics.

It's worthwhile to point out that the same technique discussed in this paper can be extended to calculate the non-linear fringe maps in a quadrupole, which has been discussed in Ref. [4]. Including kinematic terms in the fringe maps is also trivial.

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