

TWO CELL REPETITIVE ACHROMATS AND FOUR CELL ACHROMATS BASED ON MIRROR SYMMETRY

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Abstract

In this article we introduce a group-theoretical point of view for the design of magnetic optical achromats based on symmetry. As examples we use two-cell repetitive achromats and four-cell achromats employing mirror symmetry.

INTRODUCTION

As an achromat we will understand a particle transport system whose linear transfer matrix is dispersion free (first-order achromat) and whose transfer map does not have nonlinearities of transverse motion up to a certain order n (n -order achromat). Although first-order achromats are known and are widely used since the mid of 1950s [1], the possibility to use second-order achromats in practical accelerator designs was considered as unrealistic for a long time. It is related to the fact that even with mid-plane symmetry taken into account the transfer map of a magnetostatic system can have as much as 18 independent transverse second-order aberrations and thus requires at least the same number of independent sextupoles (or sextupole families) for their correction. So it was somewhat a surprise when at the end of 1970s the design of a four-cell second-order achromat which uses only 8 sextupoles arranged in only two independent families was found [2]. The theory developed in that paper (theory of achromats based on repetitive symmetry) states, that any system built out of n identical cells ($n > 1$) with the overall transfer matrix equal to the identity matrix (except, possibly, for the element relating the time-of-flight difference to the energy difference) and with the tunes of a cell such that resonances which are not forbidden by the mid-plane symmetry are avoided to third order gives a first-order achromat and can be corrected to become a second-order achromat using only two families of sextupoles. Later on the theory of achromats, which employ mirror (reversal) symmetry, was also developed in [3].

Even if it is clear that automatic cancellation of some aberrations in the symmetry based achromat designs follows from the symmetry of the magnet arrangement in the system, there are several questions which are not yet answered by any of the presently known theories. What is the exact role of the symmetry of the magnet arrangement? Why is the system transfer matrix equal to the unit matrix in both transverse planes in almost all known achromats (with the exception of a magnifying magnetic optical achromat [4])? And the most important question is, is there a magnet arrangement which will give better cancellation of aberrations than those already known?

In this paper we will show a way to answer these questions by looking at the symmetry based achromat designs from the point of view of the theory of finite matrix groups. In our approach the system is a good candidate for making a second-order achromat if its aberrations, when represented in the form of a third order homogeneous polynomial entering the Lie exponential factorization of the system transfer map, form not an arbitrary polynomial but a polynomial which is an invariant under the action of some finite matrix group. As long as the transverse particle motion is uncoupled and as long as any finite subgroup of the group of two by two invertible matrices is either C_n (cyclic group of order n) or D_n (dihedral group of order $2n$), from the point of view of our theory there is no better cancellation of aberrations than provided by the action of C_n or D_n groups. Nevertheless, in our opinion it does not mean that one has to stop looking for new appearances of cyclic and dihedral groups in the magnetic systems. For example, the cyclic group shows itself not only in the repetitive achromats constructed from identical cells, but also in staircase achromats, in magnifying achromats, and probably some other useful manifestations of this group could be found.

Due to space limitation, in this paper we will consider only two-cell second-order repetitive achromats (group C_2) and four-cell second-order achromats based on mirror symmetry (group D_2). The general theory including a discussion of higher order achromats and invention of achromats based on arbitrary order dihedral group will be presented in a separate publication [5]. We have selected these two achromats not only because they are manifestations of the smallest nontrivial cyclic group and of the smallest dihedral group which is not a cyclic group, but also because they were achromats which actually motivated this investigation and which we studied first during the design of the arcs of the post-linac collimation section of the European XFEL Facility [6, 7].

MAPS AND ACHROMATS

We will consider the beam dynamics in a mid-plane symmetric magnetostatic system and will use a complete set of symplectic variables $z = (x, p_x, y, p_y, \sigma, \varepsilon)$ as particle coordinates. In this set the variables $\hat{z} = (x, p_x, y, p_y)$ describe the transverse particle motion and the variables σ and ε characterize the longitudinal dynamics. We will represent particle transport from the longitudinal location τ_1 to the location τ_2 by a symplectic map $\mathcal{M}_{1,2}$ and we will assume that for arbitrary positions τ_1, τ_2 the point $z = 0$ is the fixed point and that the map $\mathcal{M}_{1,2}$ can be Taylor expanded in its neighborhood. If a beam passes successively

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through the maps $\mathcal{M}_{1,2}$ and $\mathcal{M}_{2,3}$ then we will use the following notation for the map of the composite system¹

$$: \mathcal{M}_{1,3} := : \mathcal{M}_{2,3}(\mathcal{M}_{1,2}) := : \mathcal{M}_{1,2} :: \mathcal{M}_{2,3} : . \quad (1)$$

Up to any predefined order n the aberrations of a map \mathcal{M} can be represented through a Lie factorization as

$$: \mathcal{M} :=_n \exp(: \mathcal{F}_{n+1} + \dots + \mathcal{F}_3 :) : M : , \quad (2)$$

where each of the functions \mathcal{F}_m is a homogeneous polynomial of degree m in the variables z and the symbol $=_n$ denotes equality up to order n when maps on both sides of (2) are applied to the phase space vector z . Using this representation we will say that the map \mathcal{M} is a n -order achromat if the matrix of its linear part M is dispersion free and all polynomials \mathcal{F}_m are functions of the variable ε only.

We will use that the map \mathcal{M} of a magnetic system which is symmetric about the horizontal midplane $y = 0$ satisfies

$$: \mathcal{M} :: T_M := : T_M :: \mathcal{M} : , \quad (3)$$

where $T_M = \text{diag}(1, 1, -1, -1, 1, 1)$ is the mid-plane symmetry matrix.

TWO CELL REPETITIVE ACHROMAT AND CYCLIC GROUP OF ORDER TWO

Let us consider a system constructed by a repetition of two identical cells with the cell map \mathcal{M}_c given by the following Lie factorization

$$: \mathcal{M}_c :=_2 \exp(: \mathcal{F}_3(z) :) : M_c : . \quad (4)$$

First-Order Conditions

The necessary and sufficient conditions for a lattice made of two identical cells to be a first-order achromat are

$$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \begin{pmatrix} r_{16} \\ r_{26} \end{pmatrix} = - \begin{pmatrix} r_{16} \\ r_{26} \end{pmatrix}, \quad (5)$$

where r_{mk} are the elements of the cell matrix M_c .

Second-Order Conditions

Let us assume that the linear achromat conditions (5) are satisfied. Then it is possible to show that the equation

$$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} r_{16} \\ r_{26} \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \quad (6)$$

can always be solved with respect to the variables A and B . And though in some rare cases the solution can be non-unique, one sees that any A and B satisfying (6) give initial conditions for the cell periodic coordinate and momentum dispersion functions. This means that if condition (5) is satisfied, the matrix M_c can be written in the form

¹Since the chance of confusion is small, we will use symbol $: * :$ to denote both, map compositions and exponential Lie operators.

$$M_c = D_c N_c D_c^{-1}, \quad (7)$$

where the matrix N_c is dispersion-free and the matrix D_c can be represented in the form of a Lie operator as follows

$$: D_c := \exp(: \varepsilon (B x - A p_x) :) . \quad (8)$$

Using (7) the cell transfer map can be brought into the form

$$: \mathcal{M}_c :=_2 : D_c^{-1} \exp(: \mathcal{P}_3(z) :) : N_c :: D_c : \quad (9)$$

with $\mathcal{P}_3(z) = \mathcal{F}_3(x + A\varepsilon, p_x + B\varepsilon, y, p_y, \varepsilon)$, and for the map of the two cell system $: \mathcal{M} := : \mathcal{M}_c :: \mathcal{M}_c :$ we obtain

$$: \mathcal{M} :=_2 : D_c^{-1} : \exp(: 2 \cdot \mathcal{S}_3(z) :) : D_c :: M : , \quad (10)$$

where $M = M_c M_c$ is the system transfer matrix,

$$\mathcal{S}_3(z) = (1/2) \cdot (\mathcal{P}_3(\hat{z}, \varepsilon) + \mathcal{P}_3(M_4 \hat{z}, \varepsilon)), \quad (11)$$

and the four by four matrix M_4 is the upper left block of the six by six matrix N_c (or, equivalently, of the matrix M_c). Thus, according to the representation (10), the two cell first-order achromat will become a second-order achromat if, and only if, the function \mathcal{S}_3 will be a function of the variable ε only, i.e. if $\mathcal{S}_3(\hat{z}, \varepsilon) - \mathcal{S}_3(\mathbf{0}, \varepsilon) = 0$.

Appearance of the Cyclic Group C_2 and Role of the Mid-Plane Symmetry

For the mid-plane symmetric system the polynomial \mathcal{F}_3 can have as much as 18 nonzero monomials responsible for the independent transverse aberrations. Why should one expect that the polynomial \mathcal{S}_3 has a smaller number of them, i.e. why should one expect that the map of the two cell system has less independent second order aberrations than the cell map? No reason is seen for that in the case of an arbitrary matrix M_4 . The situation will change, if we assume that $M_4^2 = I_4$ while $M_4 \neq I_4$, where I_m is the m by m identity matrix. With this assumption the matrices I_4 and M_4 will form a finite matrix group, which is isomorphic to the group C_2 , and \mathcal{S}_3 will not be an arbitrary polynomial anymore. It becomes the result of the application of the group Reynolds (averaging) operator to the polynomial \mathcal{P}_3 and for an arbitrary \mathcal{P}_3 is a polynomial which remains invariant under the group action. As an abstract object the group C_2 is unique, but there are three different choices for the matrix M_4 in order to satisfy the group condition

$$M_4 = \begin{cases} \text{diag}(-I_2, -I_2) \\ \text{diag}(I_2, -I_2) \\ \text{diag}(-I_2, I_2) \end{cases} . \quad (12)$$

Before considering the optimal choice for the matrix M_4 from the list (12), let us discuss shortly the role of the mid-plane symmetry. The commutation relation (3) tells us that \mathcal{F}_3 in (4) is not an arbitrary polynomial, but is an invariant of another C_2 group (mid-plane symmetry group) formed by the matrices I_6 and T_M . So as a total symmetry group

of the mid-plane symmetric two-cell system one can consider the group generated by the matrix M_4 and by the four by four upper left block of the matrix T_M , and then study the action of the Reynolds operator of this group on arbitrary polynomials. Or, as we prefer, one can use a slightly different approach and utilize the fact that the Reynolds operator of the group formed by the matrix M_4 maps the set of invariants of the mid-plane symmetry group into itself.

With all these discussions one can show that as optimal choice for the matrix M_4 one can take either the first or the third line in (12). Both give the number of the remaining independent transverse aberration to be corrected by sextupole magnets equal to six. With both these choices the condition (5) will be satisfied automatically for an arbitrary cell dispersions r_{16} and r_{26} , and the transfer matrix of the two cell system will be equal to the identity matrix in both transverse planes.

FOUR CELL ACHROMAT BASED ON REFLECTION SYMMETRY AND THE KLEIN FOUR-GROUP

Following [3] let us consider a system where the forward cell is followed by a reversed cell and then this two-cell configuration is repeated once more, i.e. let us consider a system whose transfer map is given by the following relation

$$: \mathcal{M} :=: \mathcal{M}_F :: \mathcal{M}_R :: \mathcal{M}_F :: \mathcal{M}_R :. \quad (13)$$

It is clear that with $: \mathcal{M}_c :=: \mathcal{M}_F :: \mathcal{M}_R :$ this system can be treated as the two-cell system considered in the previous section and, therefore, if the first-order achromat conditions (5) are satisfied, then the matrix of the half of the system $M_c = M_R M_F$ can be represented in the form (7). The property that now the cell \mathcal{M}_c is mirror symmetric with respect to its center allows us to prove additionally that B entering formula (8) is equal to zero and therefore the matrix D_c commutes with the reversal symmetry matrix $T_R = \text{diag}(1, -1, 1, -1, -1, 1)$. That is an important fact which not only allows us to obtain the representation (15), but also allows to remove one superfluous constraint on the linear optics which was used in [3] (see details in [5, 7]).

Let us assume that the map of the forward cell \mathcal{M}_F is

$$: \mathcal{M}_F :=_2 \exp(: \mathcal{F}_3(z) :) : M_F :. \quad (14)$$

Then we can represent the map of the total system as

$$: \mathcal{M} :=_2 : D_c^{-1} : \exp(: 4 \cdot \mathcal{S}_3(z) :) : D_c :: M :, \quad (15)$$

where $M = M_c M_c$ is the system transfer matrix,

$$\mathcal{S}_3(z) = (1/4) \cdot \sum_{m=1}^4 \mathcal{P}_3(A_m \hat{z}, \varepsilon), \quad (16)$$

and the four by four matrices A_1, A_2, A_3 and A_4 are the upper left blocks of the matrices $I_6, T_R N_c, N_c$ and $T_R N_c^2$, respectively, and $\mathcal{P}_3(z) = \mathcal{F}_3(x + A\varepsilon, p_x, y, p_y, \varepsilon)$.

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Looking at (16) one sees that if the matrices A_m will form a finite matrix group, then \mathcal{S}_3 is the result of applying the Reynolds operator of this group to the polynomial \mathcal{P}_3 and, therefore, is an invariant. What kind of group could it be? One can check that while matrices A_1 and A_3 are symplectic, the matrices A_2 and A_4 are antisymplectic. Because A_1 is a unit matrix and because the product of symplectic and antisymplectic matrices is an antisymplectic matrix, it follows that in order to have any kind of a group structure the matrix A_3 should be equal to its own inverse, i.e. the equation $A_3^2 = I_4$ must be satisfied. This equation has four symplectic solutions $A_3 = \text{diag}(\pm I_2, \pm I_2)$ and the choice of any of them completely determines the remaining matrices A_m and also gives us some group structure. If we take $A_3 = \text{diag}(I_2, I_2)$, then we have two exemplars of the same C_2 group and all other choices will give us the Klein four-group (D_2 group).

Similar to the situation considered in the previous section, there are two optimal choices for the matrix A_3 with the mid-plane symmetry taken into account. Namely, one can take $A_3 = \text{diag}(-I_2, -I_2)$ or $A_3 = \text{diag}(-I_2, I_2)$ and one obtains the number of the independent transverse aberrations left to be corrected by sextupoles to be four.

We see that the second-order achromat built by using the repetition of two identical cells (FF) requires six independent sextupole families while in the second-order achromat constructed from two mirror symmetric cells (FRFR) only four sextupole families are needed. But because one has to put sextupoles into achromats in such a way that the symmetry is preserved, the minimum number of sextupoles is 16 for the symmetry FRFR and only 12 for the symmetry FF. Note that this is a general situation in the so-called "resonant case". If, in such a case, the number of independent multipole families is of concern, then it is better to use achromats based on the D_n group, and if the total number of multipole magnets has to be minimized, then achromats utilizing the C_n group perform better (see details in [5]).

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