APOCHROMATIC BEAM TRANSPORT IN DRIFT-QUADRUPOLE SYSTEMS

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Abstract

Though a straight drift-quadrupole system can not be made an achromat, there exists an example of a bend-free drift-quadrupole system which can transport certain incoming beam ellipses without introducing first-order chromatic distortions [1]. In this paper we show that such a possibility is not a rare special case, but a general property. For every drift-quadrupole system there exists an unique set of Twiss parameters, which will be transported through that system without first order chromatic distortions. Moreover, we prove that at the same time these Twiss parameters minimize the absolute values of the system chromaticities and also bring the second order effect of the betatron oscillations on the longitudinal dynamics to the minimal possible value.

INTRODUCTION

A straight drift-quadrupole system can not be designed in such a way that a particle transport through it will not depend on the difference in particle energies. Moreover, this dependence can not be removed even in first order with respect to the energy deviations. One way to overcome this problem was developed in the past and consists in introducing bending of the central trajectory by a dipole field with subsequent usage of magnetic multipoles at suitable locations for correction of chromatic aberrations.

However, it often would be desirable to produce achromatic (i.e. energy independent) focusing in a straight system, without involvement of bending magnets. Because, according to the above discussion, it is not possible if one considers transport of individual particles starting from the same initial conditions in the transverse phase space but with different energies, we will consider the dynamics of a group of particles, i.e. particle ensembles. It is clear that the dynamics of a particle ensemble can be sufficiently different from the behavior of a single particle. For example, the set of particles uniformly distributed on the unit circle in the plane remains unchanged for an external observer after an arbitrary rotation around the origin of the coordinate system, while every single particle changes its position.

So, instead of comparing the dynamics of two particles starting from the same transverse position but with different energies, we will compare the results of tracking through the system two monoenergetic particle ensembles. The transverse distributions of the particles within each ensemble are assumed to be uncoupled between horizontal and vertical degrees of freedom and, for both ensembles, are matched to the same Courant-Snyder quadratic forms. In other words, instead of the dynamics of particles we will study the propagation of functions, namely Courant-Snyder quadratic forms.

If one is interested only in the lowest order effects and because the map of a bend-free drift-quadrupole system does not have second order geometric aberrations, one can equivalently look at first order chromatic distortions of the betatron functions appearing after their transport through the system. From this point of view there are examples of drift-quadrupole beamlines for which by appropriate choice of incoming (energy independent) beta and alpha functions one can remove first (and sometimes even second) order chromatic distortion of the exit beta function. But, to the author's knowledge, it is only the paper [1] which explicitly gives an example of the focusing system for which first order chromatic distortions at the exit will vanish with appropriate choice of the entrance Twiss parameters for both, exit beta and alpha functions.

In this paper we show that such a possibility is not a rare special case, but a general property. For every drift-quadrupole system (which is not a pure drift space) there exists an unique set of Twiss parameters (apochromatic Twiss parameters), which will be transported through that system without first order chromatic distortions.[1] Moreover, we prove that at the same time these apochromatic Twiss parameters minimize the absolute values of the system chromaticities and also bring the second order effect of the betatron oscillations on the difference of the average bunch path length from the path length of the reference particle to the minimal possible value (see formula (23) below). And in the case of a Gaussian beam they also minimize the effect of the betatron oscillations on the bunch lengthening.

Note that our interest in the problem of apochromatic beam transport was stimulated by the design of the beam distribution and transport lines for the European X-Ray Free-Electron Laser (XFEL) Facility, in particular by the design of matching sections and phase shifter of the post-linac collimation system [2, 3]. One of the specifications to the design of this facility is the requirement for transport lines from linac to undulators to be able to accept bunches with different energy (up to $\pm 1.5\%$ from nominal energy) and transport them without noticeable deterioration of beam parameters. This will allow to fine-tune the FEL wavelength by changing the electron beam energy without adjusting magnet strengths and, even more, will allow to

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[1] Following [1] we have found convenient to use the term *apochromat* for such type of focusing.

scan the FEL wavelength within a bunch train by appropriate programming of the low level RF system.

MAPS AND APOCHROMATS

As usual, we will take the path length along the reference orbit τ to be the independent variable and will use a complete set of symplectic variables $\boldsymbol{z} = (x, p_x, y, p_y, \sigma, \varepsilon)$ as particle coordinates [4, 5]. Here x, y measure the transverse displacements from the ideal orbit and p_x, p_y are transverse canonical monenta scaled with the constant kinetic momentum of the reference particle p_0. The variables σ and ε which describe longitudinal dynamics are

$$\sigma = c\,\beta_0\,(t_0 - t), \qquad \varepsilon = (\mathcal{E} - \mathcal{E}_0)\,/\,(\beta_0^2\,\mathcal{E}_0), \tag{1}$$

where $\mathcal{E}_0$, β_0 and $t_0 = t_0(\tau)$ are the energy of the reference particle, its velocity in terms of the speed of light c and its arrival time at a certain position τ, respectively.

We will represent particle passage through our system by a symplectic map $\mathcal{M}$ that maps the dynamical variables $\boldsymbol{z}$ from the location $\tau = 0$ to the location $\tau = l$. We will assume that $\boldsymbol{z} = \boldsymbol{0}$ is the fixed point and that the map $\mathcal{M}$ can be Taylor expanded in its neighborhood. Additionally we will assume that the transverse motion is dispersion free (always true for the drift-quadrupole systems) and uncoupled in linear approximation, which is a restriction on the form of the six by six symplectic matrix M of the linear part of our map.

Let g_0 be some function of the variables $\boldsymbol{z}$ given at the system entrance. Then its image g_l at the system exit under the action of the map $\mathcal{M}$ is given by the following relation

$$\forall \boldsymbol{z} \qquad g_l(\boldsymbol{z}) = g_0(\mathcal{M}^{-1}(\boldsymbol{z})), \tag{2}$$

which symbolically we will write as $g_l =: \mathcal{M} :^{-1} g_0$.

Let us consider some Courant-Snyder quadratic forms

$$\left\{ \begin{array}{lcl} I_x & = & \gamma_x\, x^2 + 2\alpha_x\, x\, p_x + \beta_x\, p_x^2 \\ I_y & = & \gamma_y\, y^2 \;+ 2\alpha_y\, y\, p_y + \beta_y\, p_y^2 \end{array} \right. \tag{3}$$

given at the system entrance. We will say that the map $\mathcal{M}$ is an n-order ($n \geq 2$) apochromat with respect to the incoming Courant-Snyder quadratic forms I_x and I_y if

$$: \mathcal{M} :^{-1} I_{x,y} - \; : M :^{-1} I_{x,y} \; = \; O\left(|\boldsymbol{z}|^{n+2}\right). \tag{4}$$

We will call the Twiss parameters that enter the Courant-Snyder quadratic forms satisfying (4) apochromatic Twiss parameters.

Note that though in this paper we will discuss second-order apochromats only, we have the design of a drift-quadrupole system which is a third-order apochromat and we will present this design in a separate publication.

Up to any predefined order n the aberrations of the map $\mathcal{M}$ can be represented through a Lie factorization as

$$: \mathcal{M} :=_n \exp(: \mathcal{F}_{n+1} + \ldots + \mathcal{F}_3 :) : M :, \tag{5}$$

where $=_n$ denotes equality up to order n and each of the functions $\mathcal{F}_m$ is a homogeneous polynomial of degree m in the variables $\boldsymbol{z}$.

Using this representation we can state that the map $\mathcal{M}$ will be n-order apochromat with respect to I_x and I_y if, and only if, all homogeneous polynomials $\mathcal{F}_m$ in (5) can be expressed as functions of I_x, I_y and ε only.

EXISTENCE AND UNIQUES OF APOCHROMATIC TWISS PARAMETERS

The Hamiltonian of the drift-quadrupole system expanded up to third order in the variables $\boldsymbol{z}$ then takes the form $H =_3 H_2 + H_3$, where

$$H_2 = (1/2)\big(p_x^2 + p_y^2 + \varepsilon^2/\gamma_0^2\big) + (k_1/2)\big(x^2 - y^2\big), \tag{6}$$

$$H_3 = -(\varepsilon/2)\big(p_x^2 + p_y^2 + \varepsilon^2/\gamma_0^2\big) \tag{7}$$

and $k_1 = k_1(\tau)$ is the quadrupole coefficient.

Let $M(\tau)$ be a fundamental matrix solution of the linearized system with the elements r_{mk} driven by Hamiltonian H_2. Then $\mathcal{F}_3$ entering formula (5) can be found as

$$\mathcal{F}_3(\boldsymbol{z}) = -\int_0^l H_3(\tau, M(\tau)\boldsymbol{z})\, d\tau$$

$$= -(\varepsilon\,/\,2)\cdot\big(\mathcal{Q}_x(x,p_x) \,+\, \mathcal{Q}_y(y,p_y) \,-\, l\,\varepsilon^2\,/\,\gamma_0^2\big)\,, \tag{8}$$

where $\mathcal{Q}_x$ and $\mathcal{Q}_y$ are quadratic forms.

From (8) one sees that the map of the drift-quadrupole system does not have second order geometric aberrations and that the transverse motion still remains uncoupled with first nonlinear correction terms taken into account. So we will restrict our further consideration to the motion in one degree of freedom (horizontal). Let us denote

$$\mathcal{Q}_x(x,p_x) = c_{20}\, x^2 \,+\, 2c_{11}\, x p_x \,+\, c_{02}\, p_x^2. \tag{9}$$

The coefficients of this form are given by the integrals

$$c_{20} = -\int_0^l r_{21}^2(\tau)\, d\tau, \qquad c_{02} = -\int_0^l r_{22}^2(\tau)\, d\tau, \tag{10}$$

$$c_{11} = -\int_0^l r_{21}(\tau)\, r_{22}(\tau)\, d\tau \tag{11}$$

and therefore satisfy Cauchy-Bunyakovsky inequality

$$c_{20}\, c_{02} \,-\, c_{11}^2 \,\geq\, 0, \tag{12}$$

where, as it is not difficult to prove, the equality holds if, and only if, $k_1(\tau) \equiv 0$, i.e. if our system is the pure drift space. So if $k_1(\tau) \not\equiv 0$ then the quadratic form $\mathcal{Q}_x$ is negative-definite and therefore, according to the statement at the end of the previous section, there exist unique apochromatic Twiss parameters given by the following expressions

$$\beta_x = -\frac{c_{02}}{\sqrt{c_{20}c_{02} - c_{11}^2}}, \quad \alpha_x = -\frac{c_{11}}{\sqrt{c_{20}c_{02} - c_{11}^2}}. \tag{13}$$

CHROMATIC VARIABLES

The quantities which are usually used in accelerator physics for the description of the first-order chromatic effects are chromaticity and two betatron amplitude difference functions [6]. It is intuitively clear that all of them must be connected with each other and with the chromatic map aberrations. Unfortunately the usual ways of their introduction are formally different and do not allow immediately to see such a connection. In this section we introduce chromatic variables as coefficients of the expansion of the quadratic form (9) with respect to three independent quadratic invariants of linear motion. One of these variables coincides with the usual chromaticity and two others (apochromaticities) can be expressed through betatron amplitude difference functions using rotation and scaling.

Invariants of Uncoupled Linear Motion

It is well known that for the linear dynamics the Courant-Snyder quadratic form is an invariant of motion. It is much less known that in the linear case there are other invariants which depend on betatron phase and together with the Courant-Snyder invariant form a basis in the space of polynomials [7]. For the purpose of this paper we need only two of them given by the following formulas

$$\left\{ \begin{array}{ll} U_x & = \bar{U}_x \cos(2\mu_x) - \bar{V}_x \sin(2\mu_x) \\ V_x & = \bar{U}_x \sin(2\mu_x) + \bar{V}_x \cos(2\mu_x) \end{array} \right. \tag{14}$$

where

$$\bar{U}_x = (2/\beta_x)\, x^2 - I_x, \quad \bar{V}_x = 2\big((\alpha_x/\beta_x)\, x^2 + x p_x\big). \tag{15}$$

Chromaticity and Apochromaticities

We define chromaticity ξ_x and apochromaticities ζ_x and η_x as coefficients in the representation

$$\mathcal{Q}_x(x, p_x) = \xi_x\, I_x(0) \,+\, \zeta_x\, U_x(0) \,+\, \eta_x\, V_x(0). \tag{16}$$

Comparing (16) with (9) we obtain

$$\left\{ \begin{array}{l} \xi_x = (1/2)(\beta_x(0) c_{20} - 2\alpha_x(0) c_{11} + \gamma_x(0) c_{02}) \\ \zeta_x = \xi_x - c_{02}/\beta_x(0) \\ \eta_x = c_{11} - (\alpha_x(0)/\beta_x(0))\, c_{02} \end{array} \right. \tag{17}$$

From these formulas it is not difficult to derive the following important equality

$$\xi_x^2 \,=\, \zeta_x^2 \,+\, \eta_x^2 \,+\, \left(c_{20}\, c_{02} \,-\, c_{11}^2\right). \tag{18}$$

With the representation (16) the image of the Courant-Snyder quadratic form after passage through the system is

$$:\mathcal{M}:^{-1} I_x(0) =_3 I_x(l) + 2\varepsilon\, (\zeta_x V_x(l) - \eta_x\, U_x(l))\,, \tag{19}$$

and one sees that this quadratic form will be apochromatic if, and only if, $\zeta_x = \eta_x = 0$. This together with (18) tells us that apochromatic incoming Twiss parameters simultaneously are the Twiss parameters which minimize the absolute value of the system chromaticity.

Note that for drift-quadrupole systems chromatic variables can be represented also in the form of the integrals

$$\xi_x \,=\, -\frac{1}{2} \int_0^l \gamma_x \, d\tau, \tag{20}$$

$$\zeta_x = \frac{1}{2} \int_0^l \left[\frac{1-\alpha_x^2}{\beta_x} \cos(2\mu_x) - \frac{2\alpha_x}{\beta_x} \sin(2\mu_x) \right] d\tau, \tag{21}$$

$$\eta_x = \frac{1}{2} \int_0^l \left[\frac{1-\alpha_x^2}{\beta_x} \sin(2\mu_x) + \frac{2\alpha_x}{\beta_x} \cos(2\mu_x) \right] d\tau. \tag{22}$$

EFFECT OF BETATRON OSCILLATIONS ON LONGITUDINAL MOTION

To see more clearly the effect of the betatron oscillations on the longitudinal motion, let us assume that the beam at the system entrance is monoenergetic with $\varepsilon = 0$ for all particles. If the beam distribution is uncoupled between the degrees of freedom then

$$\big\langle \sigma(l) \big\rangle =_2 \big\langle \sigma(0) \big\rangle \,+\, \epsilon_x\, \xi_x \,+\, \epsilon_y\, \xi_y, \tag{23}$$

where ϵ_x and ϵ_y are the non-normalized rms emittances and the chromaticities ξ_x and ξ_y are calculated using the beam Twiss parameters. If, additionally, we will assume that the particle distributions in each transverse plane are Gaussian, then we can also obtain the following formula for the effect of the betatron oscillations on the bunch lengthening

$$\left\langle \left[\sigma(l) - \langle \sigma(l) \rangle\right]^2 \right\rangle =_4 \left\langle \left[\sigma(0) - \langle \sigma(0) \rangle\right]^2 \right\rangle$$

$$+\, \epsilon_x^2 \cdot \left(\xi_x^2 \,+\, \zeta_x^2 \,+\, \eta_x^2\right) \,+\, \epsilon_y^2 \cdot \left(\xi_y^2 \,+\, \zeta_y^2 \,+\, \eta_y^2\right). \tag{24}$$

So one sees that the effect of the betatron oscillations on the longitudinal motion becomes minimal for the beam matched to the apochromatic Twiss parameters.

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