NONLINEAR THEORY OF WAKEFIELD EXCITATION IN A RECTANGULAR MULTIZONE DIELECTRIC RESONATOR

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Abstract

The nonlinear self-consistent theory of wake field excitation in the multilayered dielectric resonators is built. Expressions for excited fields, functionally depending on position of bunch particles in the resonator are found analytically. Excited fields are presented in the form of superposition solenoidal (LSE and LSM types) and potential fields. The nonlinear theory built in a general view is valid for any number of dielectric layers.

INTRODUCTION

Multizone dielectric structures are attractive for the use in perspective two-beam accelerators of charged particles [1,2]. The transport channel for drive bunch and the accelerating channel for witness bunch are spaced, therefore, it is possible to obtain the great value of a transformer ratio and, thus, to reach ultrahigh rate of acceleration.

Theoretical description of wake field excitation in the dielectric structures of a rectangular configuration is usually restricted to the linear approach [3], with the rigid motion of exciting bunches. Completely numerical methods [2] allow considering bunch dynamics. But applicability these methods is restricted frequently by opportunities of the up-to-date computers. Especially this problem is topical when simulating of wake fields of the terahertz frequency band.

In the present paper we report the combined method of the account of bunch dynamics on wake field excitation. Expressions for energized fields are found by analytical methods. They functionally depend from particle locations at the any time (a nonlinear source function). Together with motion equations they allow to describe self-consistently a dynamics of fields and bunches.

STATEMENT OF THE PROBLEM

The multizone dielectric structure under investigation is a rectangular metal resonator with dielectric slabs disposed parallelly to one of walls [3]. We shall direct y-axis of the cartesian coordinate system parallelly to slabs, x-axis is perpendicular to slabs. The dielectric slabs have generally various values of a permittivity $\varepsilon=\varepsilon_i\,(1\le i\le N)$ and a permeability $\mu=\mu_i\,(1\le i\le N)$, N - number of zones of the dielectric structure. In one of vacuum zone ($\varepsilon=1,\ \mu=1$), parallelly z-axis the electron bunch exciting the resonator travels. Width of the resonator is a, height is b, length is L.

Let's express required fields in the form of the total of solenoidal and potential components[4]:

$$\mathbf{E}=\mathbf{E}^{\mathbf{t}}+\mathbf{E}^{\mathbf{l}},\ \ \mathbf{H}=\mathbf{H}^{\mathbf{t}}, \qquad (1)$$

where vortex components of an electromagnetic field $\mathbf{E}^{\mathbf{t}}$ и $\mathbf{H}^{\mathbf{t}}$ ($\mathrm{div}(\varepsilon\mathbf{E}^{\mathbf{t}})=0,\ \mathrm{div}(\mu\mathbf{H}^{\mathbf{t}})=0$) satisfy to the first and the second Maxwell equations

$$\mathrm{rot}\mathbf{H}^{\mathbf{t}}=\frac{\varepsilon(x)}{c}\frac{\partial\mathbf{E}^{\mathbf{t}}}{\partial t}+\frac{4\pi}{c}\mathbf{j},\ \ \mathrm{rot}\mathbf{E}^{\mathbf{t}}=-\frac{\mu(x)}{c}\frac{\partial\mathbf{H}^{\mathbf{t}}}{\partial t}, \qquad (2)$$

and the potential electric field $\mathbf{E}^{\mathbf{l}}$ satisfies to the Gauss law

$$\mathrm{div}(\varepsilon\mathbf{E}^{\mathbf{l}})=4\pi\rho\,. \qquad (3)$$

The vortex $\mathbf{E}^{\mathbf{t}}$ and potential $\mathbf{E}^{\mathbf{l}}$ electric fields are mutually orthogonal [4] and satisfy to the boundary conditions vanishing their tangential components at metal walls of the resonator:

$$\mathbf{E}^{\mathbf{l}}_{\tau}(\mathbf{r}\in S_0)=0,\quad \mathbf{E}^{\mathbf{t}}_{\tau}(\mathbf{r}\in S_0)=0\,, \qquad (4)$$

where S_0 is designates a metal surface of the resonator, and subscript τ notates tangential component of fields..

Electron bunches will be described with macroparticles, therefore a charge density ρ and a current density $\mathbf{j}$ we shall present as follow

$$\rho=\sum_{p\in V_R}q_p\delta\left[\mathbf{r}-\mathbf{r}_p(t)\right],\ \ \mathbf{j}=\sum_{p\in V_R}q_p\mathbf{v}_p(t)\delta\left[\mathbf{r}-\mathbf{r}_p(t)\right], \qquad (5)$$

where q_p is a charge of a macroparticle, $\mathbf{r}_p$ и $\mathbf{v}_p$ are its coordinates and velocity, time-dependent. Summation in the eq. (5) is carried out on the particles which are being volume of the resonator V_R

Self-consistent dynamics of bunch particles is described by the relativistic motion equations in the electromagnetic fields excited by bunches

$$\frac{d\mathbf{p}_p}{dt}=q_p\left(\mathbf{E}+\frac{1}{m_pc\gamma_p}\mathbf{p}_p\times\mathbf{B}\right),\ \ \frac{d\mathbf{r}_p}{dt}=\frac{\mathbf{p}_p}{m_p\gamma_p}, \qquad (6)$$

where $\gamma_p^2=1+\left(\mathbf{p}_p/m_pc\right)^2$.

GREEN FUNCTION OF THE PROBLEM

Let's derive analytical solutions of set of Eqs. (1)-(5), that will allow avoiding numerical solution of them on a spatial grid. It is essential difference from numerical algorithms with use particle-in-cell (PIC) method.

POTENTIAL FIELD

Taking into account Eqs. (3) and (5), the finding of a potential electric field $\mathbf{E}^{\mathbf{l}}=-\nabla\Phi$ is reduced to the solution of the Poisson equation

$$\frac{1}{\varepsilon}\frac{\partial}{\partial x}\left(\varepsilon\frac{\partial\Phi}{\partial x}\right)+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=-\frac{4\pi}{\varepsilon}\sum_{p\in V_R}q_p\delta\left[\mathbf{r}-\mathbf{r}_p(t)\right] \qquad (7)$$

The solution of the equation (7) together with boundary conditions (4) we shall search by the decomposition method into eigenfunctions. Also the solution of the equation must satisfy to the boundary conditions consisting in continuity of potential and transverse component of vector electric induction. Using a matrix method for a finding of eigenfunctions in a multizone waveguide [3] we obtain the solution of the equation (7)

$$\Phi(x,y,z)=\frac{32\pi}{abL}\sum_{p\in V_R}\sum_{mnl}q_p\frac{\mathrm{X}_m(x_p)}{\|X_m\|^2\lambda_{mnl}}, \quad (8)$$

$$\sin k_y^n(y_p+b/2)\sin k_z^l z_p \mathrm{X}_m(x)\sin k_y^n(y+b/2)\sin k_z^l z$$

where $\lambda_{mnl}=(k_x^m)^2+\kappa_{nl}^2$, $\kappa_{nl}^2=(k_y^n)^2+(k_z^l)^2$, $k_y^n=\pi n/b$, $k_z^l=\pi l/L$.

Eigenvalues k_x^m can be determined from the solution of the dispersion equation

$$\begin{bmatrix}\cos(k_x w_1) & -\frac{1}{\varepsilon_1 k_x}\sin(k_x w_1)\end{bmatrix}\left(\prod_{j=2}^{N-1\geq 2}V^{(j)}\right)\times\begin{pmatrix}-\sin(k_x w_N)\\ \varepsilon_N k_x\cos(k_x w_N)\end{pmatrix}=0, \quad (9)$$

$$V^{(i)}=\begin{pmatrix}\cos(k_x w_i) & -\frac{1}{\varepsilon_i k_x}\sin(k_x w_i)\\ \varepsilon_i k_x\sin(k_x w_i) & \cos(k_x w_i)\end{pmatrix}, \quad (10)$$

$w_i=a_i-a_{i-1}$ is the i th zone width, a_i is right boundary and a_{i-1} is left boundary of the the i th zone ($x=a_i$, $i=0,1,...,N$). For each eigenvalue k_x there is the eigenfunction $X_m(x)=\varphi_m^{(i)}(x)$:

$$\varphi_m^{(i)}(x)=\begin{bmatrix}\cos k_x^m(x-a_i) & \frac{1}{\varepsilon_i k_x}\sin k_x^m(x-a_i)\end{bmatrix}\chi_m^{(i)}, \quad (11)$$

$$\chi_m^{(i)}=\left(\prod_{j=i+1}^{N-1\geq 2}V^{(j)}\right)\begin{pmatrix}-\sin(k_x w_N)\\ \varepsilon_N k_x\cos(k_x w_N)\end{pmatrix},\ i\leq N-2,$$
$$\chi_m^{(N-1)}=\begin{pmatrix}-\sin(k_x w_N)\\ \varepsilon_N k_x\cos(k_x w_N)\end{pmatrix},\ \chi_m^{(N)}=\begin{pmatrix}0\\ \varepsilon_N k_x\end{pmatrix}. \quad (12)$$

Functions $X_m(x)$ are orthogonal with a weight ε

$$\frac{2}{w}\int_{a_0}^{a_N}\varepsilon(x)X_m(x)X_{m'}(x)dx=\|X_m\|^2\delta_{mm'},$$
$$\|X_m\|^2=\frac{2}{w}\sum_{i=1}^{N}\varepsilon_i\int_{a_{i-1}}^{a_i}[\varphi_m^{(i)}(x)]^2dx,\quad w=\sum_{i=1}^{N}w_i. \quad (13)$$

VORTEX FIELD

For a finding of solenoidal parts of an electromagnetic field we shall solve the equations (2) with using of decomposition method on solenoidal fields of the empty multizone resonator

$$\mathbf{E}^{\mathbf{t}}=\sum_s A_s(t)\mathbf{E}_s,\ \mathbf{H}^{\mathbf{t}}=-i\sum_s B_s(t)\mathbf{H}_s, \quad (14)$$

here eigenfunchions $\mathbf{E}_s$ и $\mathbf{H}_s$ meet equations [4]

$$\mathrm{rot}\mathbf{H}_s=-ik_s\varepsilon\mathbf{E}_s,\ \mathrm{rot}\mathbf{E}_s=ik_s\mu\mathbf{H}_s, \quad (15)$$

$k_s=\omega_s/c$, and ω_s are eigenfrequencies of the dielectric resonator (indices s in Eqs. (14) and (15) substitutes three indices m, n and l).

Using the procedure developed in Ref. [4] for finding of fields excited by an exterior monochromatic source and the orthogonality condition

$$\int_{V_R}\varepsilon\mathbf{E}_s\mathbf{E}_{s'}^*dV=\int_{V_R}\mu\mathbf{H}_s\mathbf{H}_{s'}^*dV=4\pi N_s\delta_{ss'}, \quad (16)$$

in case of a non-stationary source for a finding of amplitudes of fields A_s also B_s we obtain the equations:

$$\frac{d^2A_s}{dt^2}+\omega_s^2A_s=-\frac{dR_s}{dt},\ \frac{d^2B_s}{dt^2}+\omega_s^2B_s=-\omega_sR_s, \quad (17)$$

where $$R_s=\frac{1}{N_s}\sum_{p\in V_R}q_p\mathbf{v}_p(t)\mathbf{E}_s^*[\mathbf{r}_p(t)]. \quad (18)$$

Solutions of the equations (17) look like:

$$B_s=-\int_0^t dt'\sin\omega_s(t-t')R_s(t'),\ A_s=\frac{1}{\omega_s}\frac{dB_s}{dt}. \quad (19)$$

It is known [3,5], that all components of an electromagnetic field of eigenwaves of multizone dielectric structure can be expressed through the two componets, which are perpendicular to the dielectric slabs. For the LSM waves they are expressed with using E_{xs}, and for the LSE waves they can be obtained with using through H_{xs}.

Taking into account boundary conditions (4) we write down field components for the LSM waves in the form of:

$$E_{xs}=e_{xs}^{(i)}(x)\sin k_y^n(y+b/2)\sin k_z^l z,$$
$$E_{ys}=\frac{1}{\kappa_{nl}^2\varepsilon}\frac{\partial^2}{\partial y\partial x}(\varepsilon E_{xs}),\ E_{zs}=\frac{1}{\kappa_{nl}^2\varepsilon}\frac{\partial^2}{\partial z\partial x}(\varepsilon E_{xs}), \quad (20)$$
$$H_{xs}=0,\ H_{ys}=\frac{-ik_s}{\kappa_{nl}^2}\frac{\partial}{\partial z}(\varepsilon E_{xs}),\ H_{zs}=\frac{ik_s}{\kappa_{nl}^2}\frac{\partial}{\partial y}(\varepsilon E_{xs}).$$

The traverse structure of the E_{xs} field component of the LSM wave is described by function $e_{xs}^{(i)}(x)$. This function can be found by the matrix method [3], applied for multizone waveguide case. For using of the expressions obtained in the Ref. [3] in a resonator case it is possible to replace simply the continuous longitudinal wave number k_z with its discrete values k_z^l. As a result we obatain

$$e_{xs}^{(i)}=\frac{1}{\varepsilon_i}\left(\cos k_{xs}^i(a_i-x),-\frac{\varepsilon_i}{k_{xs}^i}\sin k_{xs}^i(a_i-x)\right)\xi_s^{(i)}, \quad (21)$$

$$\xi_s^{(i)}=\left(\prod_{j=i+1}^{N-1}S_s^{(j)}\right)\begin{pmatrix}\cos(k_{xs}^N w_N)\\ \frac{k_{xs}^N}{\varepsilon_N}\sin(k_{xs}^N w_N)\end{pmatrix}A_s^{(N)}\ (i\leq N-2),$$

$$\xi_s^{(N-1)}=\begin{pmatrix}\cos(k_{xs}^N w_N)\\ \frac{k_{xs}^N}{\varepsilon_N}\sin(k_{xs}^N w_N)\end{pmatrix}A_s^{(N)},\ \xi_s^{(N)}=\begin{pmatrix}1\\0\end{pmatrix}A_s^{(N)}, \quad (22)$$

where $A_s^{(N)}$ are arbitrary constants, $k_{xs}^i = k_x^i(\omega_s)$, $\left(k_x^i\right)^2 = \omega^2\varepsilon_i\mu_i / c^2 - (k_y^n)^2 - (k_z^l)^2$, $S_s^{(i)} = S^{(i)}(\omega_s)$ and transition matrix $S^{(i)}$ for the LSM wave is defined as follows

$$S^{(i)} \equiv \begin{pmatrix} \cos k_x^i w_i & -\dfrac{\varepsilon_i}{k_x^i}\sin k_x^i w_i \\ \dfrac{k_x^i}{\varepsilon_i}\sin k_x^i w_i & \cos k_x^i w_i \end{pmatrix}. \qquad (22)$$

Eigenfrequencies $\omega_s \equiv \omega_{mnl}$ of the LSM wave are determined from the dispersion equation [3]

$$\left(\frac{k_x^1}{\varepsilon_1}\sin k_x^1 w_1, \cos k_x^1 w_1\right)\left(\prod_{i=2}^{N\geq 3} S^{(i)}\right)\begin{pmatrix} \cos k_x^N w_N \\ \dfrac{k_x^N}{\varepsilon_N}\sin k_x^N w_N \end{pmatrix} = 0 \,. \quad (24)$$

From the equations (15) for electromagnetic field components of the LSE wave with accounting of boundary conditions we obtain the expressions

$$H_{xs} = h_{xs}^{(i)}(x)\cos k_y^n(y+b/2)\cos k_z^l z,$$

$$H_{ys} = \frac{1}{\kappa_{nl}^2\mu}\frac{\partial^2}{\partial y\partial x}(\mu H_{xs}),\ H_{zs} = \frac{1}{\kappa_{nl}^2\mu}\frac{\partial^2}{\partial z\partial x}(\mu E_{xs}), \qquad (25)$$

$$E_{xs} = 0,\ E_{ys} = \frac{ik_s}{\kappa_{nl}^2}\frac{\partial}{\partial z}(\mu H_{xs}),\ E_{zs} = \frac{-ik_s}{\kappa_{nl}^2}\frac{\partial}{\partial y}(\mu H_{xs}).$$

The transverse structure of the H_{xs} field component of the LSE wave is described with function $h_{xs}^{(i)}(x)$ having in resonator case the next expression

$$h_{xs}^{(i)} = \frac{1}{\mu_i}\left(\cos k_{xs}^i(a_i - x), -\frac{\mu_i}{k_{xs}^i}\sin k_{xs}^i(a_i - x)\right)\zeta_s^{(i)}, \quad (26)$$

$$\zeta_s^{(i)} = \left(\prod_{j=i+1}^{N-1} T_s^{(j)}\right)\begin{pmatrix} -\dfrac{\mu_N}{k_{xs}^N}\sin k_{xs}^N w_N \\ \cos k_{xs}^N w_N \end{pmatrix} D_s^{(N)}\ (i \leq N-2),$$

$$\zeta_s^{(N-1)} = \begin{pmatrix} -\dfrac{\mu_N}{k_{xs}^N}\sin k_{xs}^N w_N \\ \cos k_{xs}^N w_N \end{pmatrix} D_s^{(N)},\ \zeta_s^{(N)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} D_s^{(N)}, \qquad (27)$$

$D_s^{(N)}$ are arbitrary constants, $T_s^{(i)} = T^{(i)}(\omega_s)$ and the transition matrix $T^{(i)}$ for the LSE wave is defined as follows

$$T^{(i)} \equiv \begin{pmatrix} \cos k_x^i w_i & -\dfrac{\mu_i}{k_x^i}\sin k_x^i w_i \\ \dfrac{k_x^i}{\mu_i}\sin k_x^i w_i & \cos k_x^i w_i \end{pmatrix}. \qquad (28)$$

Eigen frequencies $\omega_s \equiv \omega_{mnl}$ of the LSE wave are determined from the dispersion equation

$$\left(\cos k_x^1 w_1, -\frac{\mu_1}{k_x^1}\sin k_x^1 w_1\right)\left(\prod_{i=2}^{N\geq 3} T^{(j)}\right)\begin{pmatrix} -\dfrac{\mu_N}{k_x^N}\sin k_x^N w_N \\ \cos k_x^N w_N \end{pmatrix} = 0, \qquad (29)$$

Let's write down now the norm of a field N_s defined by expressions (16) with use of eigenfunctions $e_{xs}^{(i)}(x)$ and $h_{xs}^{(i)}(x)$.

For the LSM wave it is more convenient to define the norm using components of a magnetic field. Having substituted H_{ys} and H_{zs} from Eq. (20) in the second definition of the norm (16), we obtain

$$N_s = \frac{k_s^2 bL}{16\pi\kappa_{nl}^2}\sum_{i=1}^{N}\varepsilon_i^2\mu_i\int_{a_{i-1}}^{a_i}\left(e_{xs}^{(i)}\right)^2 dx \,. \qquad (30)$$

For the LSE wave it is more convenient to define the norm using components of a electric field. Having substituted E_{ys} and E_{zs} from Eq. (25) in the second definition of the norm (16), we obtain

$$N_s = \delta_{l,0}\frac{k_s^2 bL}{16\pi\kappa_{nl}^2}\sum_{i=1}^{N}\varepsilon_i\mu_i^2\int_{a_{i-1}}^{a_i}\left(h_{xs}^{(i)}\right)^2 dx,\ \delta_{l,0} = \begin{cases} 2, l = 0 \\ 1, l \neq 0 \end{cases}, \quad (31)$$

I.e. norm of the LSE wave is defined with use only single component of magnetic field, perpendicular to the dielectric slabs.

CONCLUSION

The set of the equations derived in the paper describes a self-consistent dynamics of relativistic electron bunches in the multizone dielectric resonator.

Dynamics of bunches is described by motion equations for macroparticles where electromagnetic fields are specified by superposition of source functions in which sources are moving macroparticles. Analytical solutions for excited fields are presented as the total of potential field and a solenoidal field. The solenoidal field is presented in the form of decomposition on eigenfunctions of the LSM and of the LSE waves. Orthogonality conditions for these waves are found and values of norms for each kind of the wave are derived.

The constructed theory allows to account nonlinear and group velocity effects in the multizone dielectric structures by simple way.

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