NONLINEAR PROPAGATION OF LASER PULSES IN PLASMAS: A COMPARISON BETWEEN NUMERICAL AND ANALYTICAL SOLUTIONS*

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Abstract

In this work the nonlinear relativistic propagation of intense lasers in plasmas is investigated. It is known that, under appropriate conditions, the ponderomotive force associated with the laser envelope can excite large amplitude electron waves (wakefields), which can be of interest for particle acceleration schemes. Numerical solutions showing some of the possible behaviors of this system are presented and compared to analytical ones, obtained through an effective potential approach using an one-dimensional Lagrangian formalism.

INTRODUCTION

Propagation of laser pulses in plasmas is a relevant subject in many fields of application. Particularly in the field of particle acceleration, development of lasers with higher power (shorter pulses) has allowed the experimental verification [1] of the ideas proposed by Tajima and Dawson [2]. The purpose of this work is to analyze the behavior of a laser pulse propagating in a plasma, considering for this two limit-situations: a narrow pulse (with its width $w(\tau) \to 0$) and a wide one ($w(\tau) \gg 1$).
We start from the same model used by *Duda and Mori* [3], but we follow as done by *de Oliveira and Rizzato* [4],[5]: applying the variational approach in a low-dimensional model (with a gaussian *ansatz* for the laser pulse) to obtain an average Lagrangian, and then the dynamical equations for the pulse width. With these equations, we can analyze the existence and behavior of stationary solutions. Finally, we compare these analysis with the numerical solution of the laser and plasma coupled equations.

MODEL

Field Equations

We describe the laser pulse propagating in a plasma using two equations: one for the vector potential a (representing the pulse envelope), another for the density of electrons in the plasma n (representing the wakefield),

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) a = -\omega_p^2\left(1 + n - \frac{|a|^2}{2}\right) a \,, \quad (1)$$

$$\left(\frac{\partial^2}{\partial t^2} + \omega_p^2\right) n = \frac{c^2}{2}\frac{\partial^2 |a|^2}{\partial x^2} \,, \quad (2)$$

$$a \equiv \frac{eA}{mc^2} \,,\ n \equiv \frac{(n_e - n_0)}{n_0} \,,\ \omega_p^2 \equiv \frac{4\pi n_0 e^2}{m}$$

where x is the direction of the laser propagation. Rescaling $t \to \omega_p t$ and $x \to (\omega_p/c)x$ and introducing the coordinates $\xi \equiv v_g^{-1}x - t$ and $\tau \equiv x$, these equations can be rewritten as

$$2ik_0\frac{\partial a}{\partial \tau} + K\frac{\partial^2 a}{\partial \xi^2} + \left(n - \frac{|a|^2}{2}\right) a = 0, \quad (3)$$

$$\left(\frac{\partial^2}{\partial \xi^2} + 1\right) n = \frac{1}{2v_g^2}\frac{\partial^2 |a|^2}{\partial \xi^2}, \quad (4)$$

where $K \equiv (1 - 1/v_g^2) \leq 0$, as $0 \leq v_g^2 \leq 1$.

Average Lagrangian

In order to obtain an average Lagrangian, from which one can derive the relevant dynamical equations through the Euler-Lagrange prescription, we first define a scalar potential

$$\phi \equiv \left(n - \frac{1}{v_g^2}\frac{|a|^2}{2}\right), \quad (5)$$

and rescale it as $\varphi \equiv v_g\phi$ to match some coefficients of equations (3) and (4), that can be written as

$$2ik_0\frac{\partial a}{\partial \tau} + K\left(\frac{\partial^2 a}{\partial \xi^2} - \frac{|a|^2 a}{2}\right) + \frac{\varphi}{v_g}a = 0, \quad (6)$$

$$\left(\frac{\partial^2}{\partial \xi^2} + 1\right)\varphi = -\frac{1}{v_g}\frac{|a|^2}{2}. \quad (7)$$

In order to keep v_g free to assume any value, we avoid the assumption of $\omega_0 = k_0$. For this reason, our expressions and definitions are slightly different from those used by *Duda and Mori* [3]. Once that the Lagrangian which generates equations (6) and (7) has been found,

$$\mathcal{L} = ik_0\left[a^*\frac{\partial a}{\partial \tau} - a\frac{\partial a^*}{\partial \tau}\right] - K\left[\frac{\partial a}{\partial \xi}\frac{\partial a^*}{\partial \xi} + \frac{(aa^*)^2}{4}\right] - \left(\frac{\partial \varphi}{\partial \xi}\right)^2 + \varphi^2 + \frac{\varphi}{v_g}(aa^*), \quad (8)$$

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we proceed with the same technique applied by *de Oliveira and Rizzato* [5], proposing an *ansatz* for the potential vector, calculating the scalar potential, applying both in the Lagrangian and, finally, integrating over ξ. We choose the following *ansatz*

$$a(\xi,\tau) = \sqrt{\frac{P}{2\pi^{1/2}w(\tau)}} exp\left[-\frac{(\xi-\lambda(\tau))^2}{2w^2(\tau)}\right] \times$$
$$\times exp\left\{i[b(\tau)(\xi-\lambda(\tau))^2 + \delta(\tau)]\right\}, \quad (9)$$

where $P = \int_{-\infty}^{\infty} |a|^2 d\xi$ is the power of the laser, $w(\tau)$ is the width of the pulse, $\lambda(\tau)$ its centroid, $b(\tau)$ is a chirping factor and $\delta(\tau)$ is a phase.
In this work we analyze two limits of this system, the narrow pulse and wide pulse regimes. This simplify the obtention of the expressions for n or φ.

NARROW LASER PULSES

We consider narrow pulses those where their width w is such that one can neglect a quantity in comparison to its second "spatial" derivative (∂_ξ^2). So, $\partial_\xi^2 n \gg n$ is valid for this condition and equation (4) allows us to write n as

$$n = \frac{1}{v_g^2}\frac{|a|^2}{2} \Rightarrow \varphi = 0, \quad (10)$$

which can be applied on equation (3) to obtain

$$2ik_0\frac{\partial a}{\partial \tau} + K\left(\frac{\partial^2 a}{\partial \xi^2} - \frac{|a|^2 a}{2}\right) = 0. \quad (11)$$

Through the application of the Euler-Lagrange equations, one can see that equation (11) can be obtained from this Lagrangian:

$$\mathcal{L} = ik_0\left[a^*\frac{\partial a}{\partial \tau} - a\frac{\partial a^*}{\partial \tau}\right] - K\left[\frac{\partial a}{\partial \xi}\frac{\partial a^*}{\partial \xi} + \frac{(aa^*)^2}{4}\right] \quad (12)$$

Inserting the *ansatz* (9) in this Lagrangian (12), integrating it over ξ to obtain $\mathcal{L}(\xi,\tau)$ as $\mathcal{L}(\tau)$ and varying it with respect to the collective variables $w(\tau), \lambda(\tau), b(\tau)$ and $\delta(\tau)$, we obtain the following dynamical equation

$$w''(\tau) = \frac{K^2\left[16\pi + \sqrt{2\pi}Pw(\tau)\right]}{16\pi k_0^2 w^3(\tau)}, \quad (13)$$

used to calculate an effective potential

$$w'' = -\frac{\partial U_w}{\partial w} \Rightarrow U_w = \frac{K^2\left[8\pi + \sqrt{2\pi}Pw(\tau)\right]}{16\pi k_0^2 w^2(\tau)}, \quad (14)$$

which do not has a fixed point, as can be seen in figure 1. As a consequence, one se only the dispersion of the pulse (figure 2).

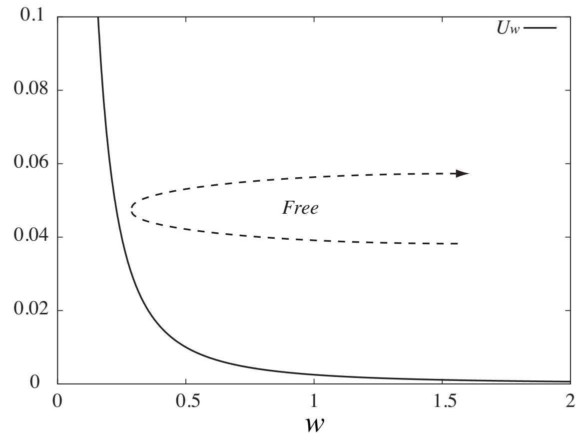

Figure 1: Effective potential fora a narrow pulse showing that there is no fixed point for the width.

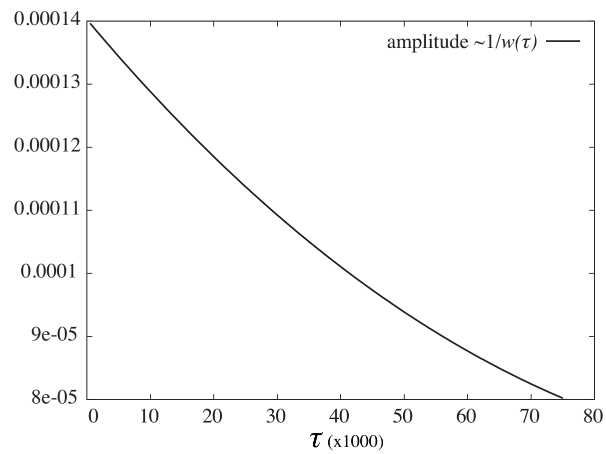

Figure 2: Numerical solution for a narrow pulse showing that for this condition there is only dispersion of the pulse.

WIDE LASER PULSES

We call here wide pulses those where their width w is such that one can neglect the second "spatial" derivative of a quantity in comparison to itself. In equation (7) for example, considering that $\partial_\xi^2 n \ll n$, we can write n as

$$n = \frac{1}{2v_g^2}\frac{\partial^2|a|^2}{\partial \xi^2} \Rightarrow \varphi = \frac{1}{2}\frac{\partial|a|^2}{\partial \xi^2} - \frac{|a|^2}{2}. \quad (15)$$

With this approximation, and also considering that (for this condition of pulse) $\partial_\xi^2|a|^2 \ll \partial_\xi^2 a$, equation (3) can be written as

$$2ik_0\frac{\partial a}{\partial \tau} + K\frac{\partial^2 a}{\partial \xi^2} - \frac{|a|^2}{2}a = 0. \quad (16)$$

This equation can be obtained applying the Euler-Lagrange equations in the following Lagrangian:

$$\mathcal{L} = ik_0\left[a^*\frac{\partial a}{\partial \tau} - a\frac{\partial a^*}{\partial \tau}\right] - K\frac{\partial a}{\partial \xi}\frac{\partial a^*}{\partial \xi} - \frac{(aa^*)^2}{4}. \quad (17)$$

Again we follow the variational approach, inserting the *ansatz* (9) in the mentioned Lagrangian (17), integrating it over ξ and, finally, varying it with respect to the collective variables $w(\tau), \lambda(\tau), b(\tau)$ and $\delta(\tau)$ to obtain (after some algebraic work) the following dynamic equation:

$$w''(\tau) = \frac{K\left[16\pi K + \sqrt{2\pi} P w(\tau)\right]}{16\pi k_0^2 w^3(\tau)}. \quad (18)$$

From this equation we can now calculate an effective potential,

$$w'' = -\frac{\partial U_w}{\partial w} \Rightarrow U_w = \frac{K\left[8\pi K + \sqrt{2\pi} P w(\tau)\right]}{16\pi k_0^2 w^2(\tau)}, \quad (19)$$

which (depending on the laser power) has a stable fixed point, as can be seen in figure 3, located at

$$w_* = -\frac{8\sqrt{2\pi}K}{P}. \quad (20)$$

Expanding (19) in the vicinity of (20) we find the linear frequency

$$\Omega_w^2 = \frac{P^4}{(128)^2 k_0^2 \pi^2 K^2} \quad (21)$$

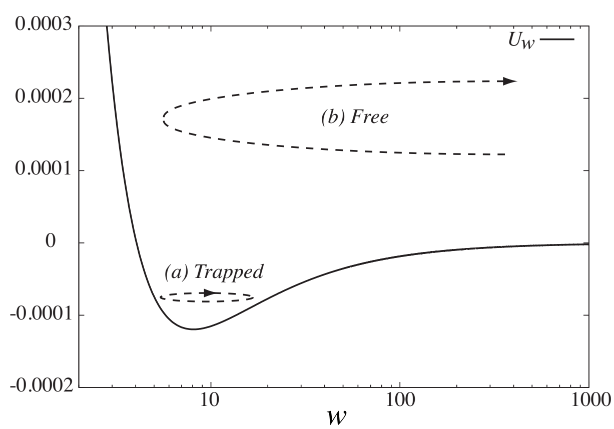

Figure 3: Effective potential showing that *(a)* for lower values of energy the width $w(\tau)$ can be trapped, resulting in soliton-like solutions oscillating around the stable fixed point, and *(b)* for higher energies the width is not confined and only dispersion of the pulse is observed.

CONCLUSIONS

In the present work we analyzed the propagation of a laser pulse in a plasma in two distinct limit situations (narrow and wide pulses), through the use of the variational approach to obtain the dynamical equations for its width. Results showed that for wide pulses there is no stationary solution. For this reason, while this condition is valid, one will see only a dispersive behavior in the laser pulse. For

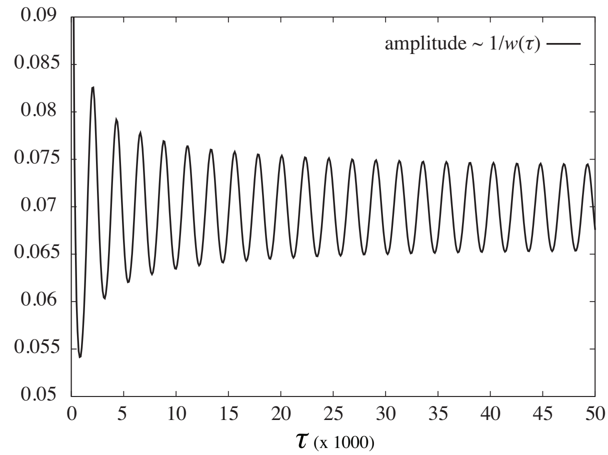

Figure 4: Numerical solution for a wide pulse. A small perturbation in the equilibrium solution of $w(\tau)$ was done to show that it is trapped, oscillating around its fixed point.

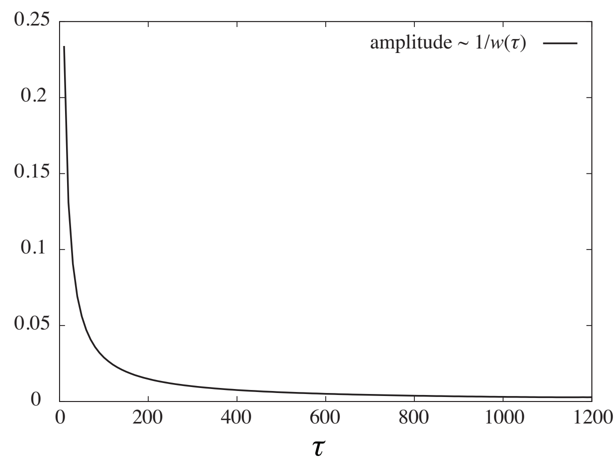

Figure 5: Numerical solution for a wide pulse with higher energy, showing that for this condition there is only dispersion.

wide pulses, a stable fixed point can exist, depending on the energy of the pulse. This behavior can be understood looking to the effective potential: if the energy is low enough, $w(\tau)$ is trapped and we have a soliton-like solution oscillating around the fixed point (figure 4). For higher values of energy, the width is not confined and $w(\tau) \to \infty$ (figure 5).

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