ANOMALOUS DIFFUSION NEAR RESONANCES *

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Abstract

Synchro-betatron resonances can lead to emittance growth and the loss of luminosity. We consider the detailed dynamics of a bunch near such a low order resonance driven by crossing angles at the collision points. We characterize the nature of diffusion and find that it is anomalous and sub-diffusive. This affects both the shape of the beam distribution and the time scales for growth. Predictions of a simplified anomalous diffusion model are compared with direct simulations.

INTRODUCTION

Transport of particles near resonances is still not a well understood phenomenon. Often, without justification, phase space motion is assumed to be a normal diffusion process although at least one case of anomalous diffusion in beam dynamics has been reported [1]. Here we will focus on the motion near synchro-betatron resonances which can be excited by several means, including beams crossing at an angle at the collision points as in the LHC. We will consider low order resonances which couple the horizontal and longitudinal planes, both for simplicity and to observe large effects over short time scales. While the tunes we consider are not practical for a collider, nonetheless the transport mechanisms we uncover are also likely to operate at higher order resonances.

AMPLITUDE DIFFUSION

We consider the four 1st and 2nd order sideband resonances around the third order horizontal resonance, i.e. the resonances $3q_x \pm q_s = n$ and $3q_x \pm 2q_s = n$. The vertical tune is fixed at 0.32. Other beam parameters have values set to design values in the LHC. In our simulation model the only nonlinearity is that due to the beam-beam interaction at two interaction points (IP) with the beams crossing at an angle in the horizontal plane at one IP and the vertical plane at the other IP. The beam-beam force vanishes at large amplitudes, hence particle motion will remain bounded but they can still be transported to the closest physical aperture such as a collimator. Since the beam-beam force is defocusing with colliding proton beams, small transverse amplitude particles will be at lower tunes than the nominal tunes. The beam-beam tune spread ($\simeq 0.007$) from the 2 IPs is larger than the synchrotron tune ($\simeq 0.002$), hence when the nominal tunes lie on the difference resonances (3,0,-1) and (3,0,-2) in the (q_x, q_y, q_s) planes, some of the particles in the distribution will straddle the 3rd order betatron resonance (3,0,0). We expect therefore that the difference resonances will have a larger impact on emittance growth and lifetime.

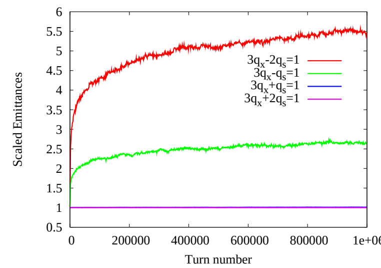

Figure 1: Growth of the horizontal emittance (scaled by the initial value) at the four resonances.

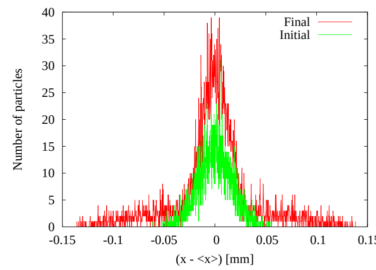

Figure 2: Initial and final beam distributions in horizontal position. The initial distributions are Gaussian. The final horizontal distribution develops long tails and is not a Gaussian.

Figure 1 shows the growth of the horizontal emittance (scaled by initial values) at these four resonances. The horizontal emittance grows more than five times for the (3,0,-2) resonance and about 2.5 times for the (3,0,-1) resonance while it is virtually unchanged for the sum resonances. The change in vertical emittance for the (3,0,-2) resonance is about 10% and there's almost no change for the other resonances.

The distributions of particle positions was studied starting with a Gaussian distribution in both planes. Figure 2 shows the initial and final (at the end of 10^6 turns) distribution in positions. We find that the final horizontal distribution has long non-Gaussian tails and the distribution develops a cusp at the center. The final vertical distribution (not shown here) stays Gaussian. Thus the synchro-betatron resonances increase the emittance and also produce long transverse tails. Normal diffusion in action is characterized by a linear growth of the variance over time. Figure 3 shows the growth of the variance in horizontal action at different amplitudes and the monomial fits. At all amplitudes, the growth is much slower than linear. The variance increase is the largest at 1.5σ and decreases at larger ampli-

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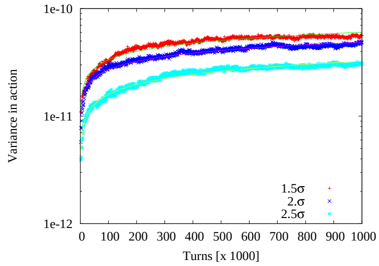

Figure 3: Variance in the actions over time at a tune corresponding to the resonance $3q_x - 2q_s = 1$. Also shown are the (barely visible) monomial fits to the data.

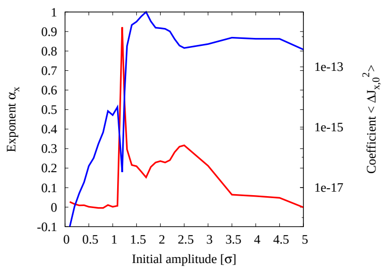

Figure 4: The exponent α_x (red, left vertical scale) and the coefficient $\Delta J^2_{x,0}$ (blue, right vertical scale) as a function of the initial amplitude.

tudes. The increase in the vertical variance is smaller by a few orders of magnitude but the growth rate is steeper. We model the growth in the variance of the actions (J_x, J_y) as: $\langle \Delta J_x^2 \rangle = \Delta J^2_{x,0} t^{\alpha_x}$, $\langle \Delta J_y^2 \rangle = \Delta J^2_{y,0} t^{\alpha_y}$ For normal diffusive behaviour, both the exponents $(\alpha_x, \alpha_y) = 1$ while anomalous sub-diffusive behaviour is characterized by exponents < 1 and super-diffusive behaviour has exponents > 1. Both exponents are < 1 at all amplitudes, but in many regions α_y is about twice α_x, yet $\langle \Delta J_y^2 \rangle$ grows much more slowly because the constant coefficient $\Delta J^2_{y,0}$ is about five orders of magnitude smaller than $\Delta J^2_{x,0}$. Figure 4 shows the exponent α_x and the constant coefficient $\langle \Delta J^2_{x,0} \rangle$ as a function of the initial amplitude. At amplitudes below 1σ and above 4σ, $\alpha_x \simeq 0$ implying no diffusion . The diffusion is fastest at 1.7σ, the location of the peak in the coefficient $\Delta J^2_{x,0}$.

We can define diffusion coefficients in the usual manner, e.g. $D_{x,x} = Var(J_x)/N$, $D_{x,y} = Covar(J_x, J_y)/N$, where N is the total number of turns. Fig 5 shows that the "diffusion coefficients" $D_{x,x}$ calculated at a few amplitudes with $x = y$ for the different resonance tunes. We do not show here the dependence of the coefficients $D_{x,x}, D_{x,y}, D_{y,y}$ on both (J_x, J_y). There is a sharp rise in these coefficients at $\sim 2.0\sigma$ for the (3,0,-1) resonance and at $\sim 1.5\sigma$ for the (3,0,-2) resonance with a larger jump for the latter. In the following we will focus on the (3,0,-2) resonance although the qualitative conclusions are applicable to the other resonances. We can (without a priori justification) use the diffusion coefficients in the normal diffusion

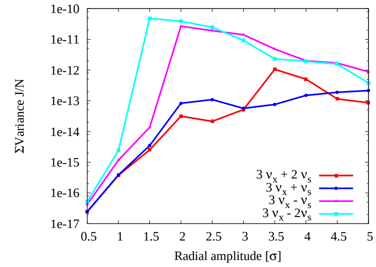

Figure 5: The horizontal "diffusion" coefficients as a function of radial amplitude at tunes corresponding to the four SBR tunes.

in action equation for the density distibution,

$$\frac{\partial}{\partial t}\rho(J_x, J_y) = \frac{1}{2}\nabla_i[D_{ij}\nabla_j]\rho(J_x, J_y) \qquad (1)$$

Here (i, j) run over (x, y) and D_{ij} are the diffusion coefficients defined above. Numerically solving this diffusion equation with the diffusion coefficients found above leads to predictions that disagree spectacularly with the direct particle tracking. For example, the solution to this diffusion equation shows that about 20% of particles are lost at a 6σ aperture within a few seconds while direct tracking shows an insignificant loss over this period. This is not surprising given that the action does not grow as rapidly as assumed by the diffusion equation. Clearly we need a different transport equation to model the diffusion process.

CTRW MODEL FOR ANOMALOUS DIFFUSION

The fact that the motion is sub-diffusive is to be expected since the persistence of the KAM tori both below and above the resonance islands will slow growth. Particles can circulate around resonance islands for long periods of time which can also lead to subdiffusion. We need to identify the most plausible model for subdiffusion applicable to our problem.

At the chosen tune of interest, the resonance islands in horizontal phase space lie at around 2σ. Motion in their vicinity but at slightly smaller amplitudes can be quite complicated. Single trajectories starting from amplitudes around 1.5σ explore much of phase space: regions well below the islands, regions around the islands as well as regions outside the islands. The motion jumps between these regions with different amplitudes, and the time spent in each region appears to be random. These are the usual ingredients needed for the continuous time random walk (CTRW) model [2] of anomalous diffusion where the time at which a step occurs is also taken to be a random variable. The CTRW model leads to a fractional diffusion equation where the order of the time derivative is fractional.

The step size distribution is one of the quantities that characterize a CTRW model. Figure 6 shows distributions in horizontal step sizes for initial amplitudes of 0.1σ and 1.55σ respectively. The left plot in this figure is typical for

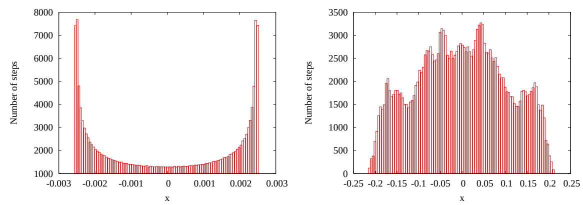

Figure 6: Step size distributions. Left: initial amplitude 0.1σ. Here motion is quasi-periodic. Right: initial amplitude 1.55σ. Here the motion is quasi-ergodic.

particles with amplitudes $< 1.5\sigma$ as well as for particles with amplitudes $> 4\sigma$. In fact this distribution appears to be universal whenever the motion is quasi-periodic, e.g. at all amplitudes and other tunes when they are far from resonances. The other distribution at 1.55σ where small step sizes are dominant, is typical at intermediate amplitudes where the motion is quasi-ergodic, in this case for amplitudes in the range $1.5\sigma < a < 4.0\sigma$. Thus the transition from quasi-periodic motion to subdiffusive behaviour and back to quasi-periodic motion is also seen in the step size distribution.

A fractional diffusion equation for the density distribution function often used to describe subdiffusive behaviour is [3]

$$\frac{\partial}{\partial t}\rho(\vec{r},t) = {}_0D_t^{1-\alpha}\nabla[K_\alpha \nabla \cdot \rho(\vec{r},t)] \tag{2}$$

Here K_α is the diffusion coefficient and the operator ${}_0D_t^{1-\alpha}$ is defined as the integro-differential operator

$${}_0D_t^{1-\alpha} = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t dt' \frac{\rho(\vec{r},t')}{(t-t')^{1-\alpha}} \tag{3}$$

In one dimension, when the diffusion coefficient K_α is constant, it can be shown [3] that the mean squared displacement is given by

$$\langle x^2(t)\rangle \equiv \int x^2 \rho(x,t)dx = \frac{2K_\alpha}{\Gamma(1+\alpha)}t^\alpha \tag{4}$$

If the exponent α is determined from a numerical calculation of $\langle x^2(t)\rangle$, then it determines the value to be used in the fractional diffusion equation. This diffusion equation in one space dimension and constant K_α can be solved in terms of the so-called Fox function $H_{1,2}^{2,0}$ which has the series representation [3]

$$\rho(x,t) = \frac{1}{\sqrt{4K_\alpha t^\alpha}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1-\alpha(n+1)/2)}\left(\frac{x^2}{K_\alpha t^\alpha}\right)^{n/2} \tag{5}$$

Fig 7 shows the density distribution function given by Eq (5) for $\alpha = 0.2$ (a typical value seen in Fig 4) and three values of t. This solution has some of the features seen in the numerical simulation, seen in Fig 2, such as the long non-Gaussian tails and the cusp at the center of the distribution. However Eq (5) does not represent the true solution since it assumes a constant diffusion coefficient K_α wbile

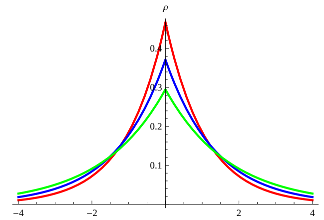

Figure 7: Density distribution function which is the solution to Eq (5) with $\alpha = 0.2$ for three different times $t = 1, 10, 100$. Compare with Fig 2.

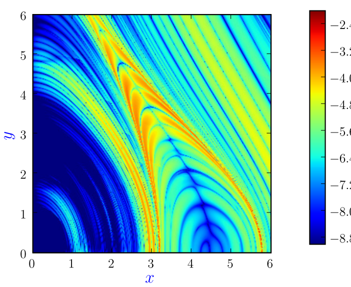

Figure 8: Frequency diffusion in the transverse plane over amplitudes from 0-6 σ at the resonance $3q_x - 2q_s = 1$.

in our case both the diffusion coefficient and power law depend on the phase space amplitude. A numerical solution of the fractional diffusion equation will be necessary to treat anomalous beam diffusion.

Finally we turn to frequency map analysis [4], another diagnostic method of detecting the transition from quasi-periodic to irregular motion. When motion is not quasi-periodic, tunes are not well defined on a phase space curve and hence are more likely to "diffuse". A figure of merit is defined as $Dq = \sqrt{(q_{x,2}-q_{x,1})^2 + (q_{y,2}-q_{y,1})^2}$ where $(q_{x,1}, q_{x,2}), (q_{y,1}, q_{y,2})$ are the tunes of a particle calculated over the two sets of N consecutive turns. Figure 8 shows a contour plot (on a log scale) of this index Dq over coordinate space. The red regions show the zones with largest "tune diffusion", this occurs mostly in the region $2 \le x \le 4$, $3 \le y \le 4$ and also in a small arc at a radial amplitude of 1.5σ. These regions roughly match the quasi-ergodic regions identified by the amplitude diffusion and the step size distribution functions.

Our goal is to understand the transport process and to predict emittance growth rates and beam lifetimes. Quasi-ergodic and subdiffusive motion in the vicinity. of resonances as found here may be understood by anomalous diffusion models but further development is necessary.

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