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Some Problems of the Beam Extraction from Circular Accelerators
The experiments using extracted beams
From physical model to mathematical models

1. The accuracy of approximation of the ideal mapping generated by the dynamical system: how estimate the closeness of ideal solutions and the corresponding approximate solutions?

2. Preserving the qualitative properties inherent in the dynamical system under study:
   – symplectic property for Hamiltonian systems;
   – conservation of exact and approximate integrals of motion and so on.

3. Constructing accurate maps for some practical classes of dynamical systems.

4. The dynamics of the beam as an ensemble.

5. The problem of parallel and distributed computing.
Physical models

Slow extraction using third-order resonance
Physical models

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Physical models

Slow extraction using third-order resonance

Inertial properties of the beam for different amplitudes of the feedback impulse
The transport channel Booster-into-Nuclotron has a three-dimensional geometry: magnetic elements do not lie in the same plane. In this case we should use the total motion equation:
- the increasing of emittance is a result of the parameters mismatch;
- the increasing of emittance is a result of the error dispersion;
- the type of particles - ions of gold with charge.
Mathematical models -1

Evolution equations (coordinates)

\[ \mathbf{v} = \frac{d \mathbf{R}}{dt}, \quad \frac{d(m \mathbf{v})}{dt} = q \left( \mathbf{E} + [\mathbf{v} \times \mathbf{B}] \right). \]

\[
B_x = \frac{\partial \psi}{\partial x}, \quad B_y = \frac{\partial \psi}{\partial y}, \quad B_z = \frac{1}{1 + hx} \frac{\partial \psi}{\partial s},
\]

\[
\psi(x, y, s) = \sum_{k,i=0}^{\infty} a_{ik}(s) \frac{x^i y^k}{i! k!}.
\]

\[
x' - nx = -h - \frac{x'(2hx' + h'x)}{1 - hx} - \frac{1}{c\beta\gamma(1 - hx)} \left( (1 - hx)^2 + x'^2 + y'^2 \right)^{1/2} \times
\]
\[
x' y' B_x - \left( (1 - hx)^2 + x'^2 \right) B_y + (1 - hx)y' B_z,
\]

\[
y'' = \frac{y'(2hx' + h'x)}{1 - hx} + \frac{1}{c\beta\gamma(1 - hx)} \left( (1 - hx)^2 + x'^2 + y'^2 \right)^{1/2} \times
\]
\[
x' y' B_y - \left( (1 - hx)^2 + y'^2 \right) B_x - (1 - hx)x' B_z.
\]

\[
a_{ik}''' + k h a_{ik}' - k h a_{ik-1}' + a_{i+2,k} + a_{i,k+2} +
\]
\[
\quad + (3k + 1) h a_{i,k+1} + k(3k - 1) h^2 a_{ik} + k(k - 1) h^2 a_{i,k-1} +
\]
\[
\quad + 3 k h a_{i+2,k-1} + 3 k(3k - 1) h^2 a_{i+2,k-2} + k(k - 1)(k - 2) h^3 a_{i+2,k-3} = 0.
\]

\[
\frac{d \mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}, t) = \mathbf{F}_{ext}(\mathbf{B}_{ext}(\mathbf{X}, t), \mathbf{E}_{ext}(\mathbf{X}, t), \mathbf{X}, t) + \mathbf{F}_{self}(\langle f(\mathbf{X}, t) \rangle_{\mathcal{M}}, \mathbf{X}, t)
\]
Mathematical models -2
Matrix Formalism - motion equations

\[
\frac{dX}{dt} = F(X, t), \quad F(0, t) \equiv 0 \quad \Rightarrow \quad \frac{dX}{dt} = \sum_{k=1}^{\infty} P^1 X[k] = \sum_{k=0}^{\infty} \frac{\partial^k F(0, t)}{\partial X^k} \frac{X^k}{(k)!}
\]

**Kronecker product**

\[
X[k] = X \otimes \ldots \otimes X, \quad k \text{-times}
\]

\[
\frac{dX[k]}{dt} = \sum_{k=0}^{\infty} X[j] \otimes \frac{dX}{dt} \otimes X[k-j-1] = \sum_{k=1}^{\infty} \sum_{j=0}^{k} X[j] \otimes P^1 \otimes X[k].
\]

\[
P^{k,j} = P^1 (j-k+1) \oplus P^{(k-1)} (j-1), \quad j \geq k,
\]

\[
P^{k,k} = P^1 \oplus P^{(k-1)} (k-1) = (P^{11})^k, \quad k \geq 2.
\]

\[
\frac{dX[\infty]}{dt} = P^{\infty}(t)X[\infty]
\]

\[
P^{\infty} =
\begin{pmatrix}
P^{11} & P^{12} & \cdots & P^{1k} \\
0 & P^{22} & \cdots & P^{2k} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & P^{2k}
\end{pmatrix}
\]
Mathematical models -3
Matrix Formalism - reverse evolutionary matrices

\[ \mathbf{X}^\infty = \mathbf{R}^\infty(t) \mathbf{X}_0^\infty \quad \overset{\leftrightarrow}{=} \quad \mathbf{X}(t) = \sum_{k=1}^{\infty} \mathbf{R}^{1k}(t|t_0) \mathbf{X}_0^{[k]} \]

\[ \mathbf{R}^{ik}(t|t_0) = \sum_{j=i+1}^{k} \int_{t_0}^{t} \mathbf{R}^{ii}(t|\tau) \mathbf{P}^{ij}(\tau) \mathbf{R}^{jk}(\tau|t_0) d\tau, \quad \mathbf{R}^{ii}(t|t_0) = (\mathbf{R}^{11}(t|t_0))^{[i]} \]

\[ (\mathbf{R}^\infty)^{-1} = \mathbf{T}^\infty = \begin{pmatrix} T^{11} & T^{12} & \cdots & T^{1k} & \cdots \\ 0 & T^{22} & \cdots & T^{2k} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \cdots & T^{kk} & \cdots \\ \vdots & \vdots & \cdots & \cdots & \cdots \end{pmatrix} \]

\[ T^{kk} = (\mathbf{R}^{kk})^{-1} = (\mathbf{R}^{11})^{-[k]} = \left( (\mathbf{R}^{11})^{-1} \right)^{[k]} \]

\[ T^{ik} = -\sum_{l=i}^{k-i} T^{il} \mathbf{R}^{lk} T^{kk}, \quad i < k. \]
Mathematical models -3
Matrix Formalism – for envelope nonlinear matrices

\[ \mathcal{G}^{ik}(t) = \int_{\mathcal{M}(t)} f(X, t) X[i] (X[k])^* dX \]

Here \( f(X, t) \) the distribution function on the set \( \mathcal{M}(t) \), occupied by particles.

\[ \frac{d\mathcal{G}^{ik}}{dt} = \sum_{l=i}^{\infty} \mathcal{P}^{il} \mathcal{G}^{lk} + \sum_{l=k}^{\infty} \mathcal{G}^{lk} (\mathcal{P}^{kl})^*, \quad i, k \leq 1 \]

\[ \frac{dX[k]}{dt} = \sum_{j=k}^{\infty} \mathcal{P}^{kj} X[j] \]

\[ \mathcal{G}^{ik}(t) = \sum_{l=i}^{\infty} \sum_{j=k}^{\infty} \mathcal{R}^{il}(t|t_0) \mathcal{G}_0^{lj} (\mathcal{R}^{kj}(t|t_0))^* \]
Mathematical models -3

The matrix presentation using Lie algebraic tools

According to the well known Lie algebraic tools\(^1\) the our motion equations can be written using so called Lie map (an evolution operator in the exponential form)

\[
\frac{dM(t|t_0)}{dt} = V(t) \circ M(t|t_0), \text{ with the initial condition } M(t_0|t_0) = \mathcal{I}d \quad \forall \ t_0 \in \mathcal{I}
\]

where

\[
V(t) = L_F = F^*(\mathbf{X}) \frac{\partial}{\partial \mathbf{X}} = \sum_{k=0}^{\infty} \left( \mathbf{X}^{[k]} \right)^* (F_k)^* \frac{\partial}{\partial \mathbf{X}} = \sum_{k=0}^{\infty} L^k_F
\]

is a Lie operator.

The solution the operator equation can be written in the form of chronological ordered series (Volterra series)

\[
M(t|t_0) = \mathcal{I}d + \sum_{k=1}^{\infty} \int_{t_0}^{t} \int_{t_0}^{\tau_1} \ldots \int_{t_0}^{\tau_{k-1}} V(\tau_k) \circ V(\tau_{k-1}) \circ \ldots \circ V(\tau_1) \, d\tau_k \ldots d\tau_1.
\]

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\(^1\) See, for example, A.J.Dragt *Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics*. University of Maryland, College Park. [www.physics.umd.edu/dsat/](http://www.physics.umd.edu/dsat/).
Mathematical models - 3

Hamiltonian formalism

In the case of Hamiltonian motion equation we can write

\[ \frac{dX}{dt} = \mathcal{J}(X) \frac{\partial \mathcal{H}(X, t)}{\partial X}, \]

where

\[ \mathcal{H} = \sum_{k=2}^{\infty} \mathcal{H}^k(t) X^{[k]} = - (1 + h x) \frac{q}{\mathcal{E}_0} A_s - \}

\[ - (1 + h x) \left[ (1 + \eta)^2 - \left( \frac{m_0 c^2}{\mathcal{E}_0} \right)^2 - \left( P_x - \frac{q}{c} A_x \right)^2 - \left( P_y - \frac{q}{c} A_y \right)^2 \right]^{1/2} \]

One can write

\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \sum_{k=3}^{\infty} \varepsilon^{k-2} \mathcal{H}_k, \]

where \( \mathcal{H}_k \) are homogeneous polynomials of \( k \)-th order. Here \( \mathcal{H}_k(t) \cdot \) vectors of coefficients for these polynomials.

This decomposition leads to expansion for motion equation of the corresponding series in according Dragt’s approach. After this one can write

\[ \mathcal{M} \left( t | t_0; \sum_{k=1}^{\infty} \mathcal{L}_{F_k} \right) = \ldots \circ \mathcal{M} \left( t | t_0; \mathcal{L}_{G_k} \right) \circ \mathcal{M} \left( t | t_0; \mathcal{L}_{G_{k-1}} \right) \circ \ldots \]

\[ \ldots \circ \mathcal{M} \left( t | t_0; \mathcal{L}_{G_2} \right) \circ \mathcal{M} \left( t | t_0; \mathcal{L}_{G_1} \right), \]
The chronological series is not convenient for practical computation. Instead of this series there is used so called Magnus presentation for Lie map

\[ M(t|t_0) = \exp \mathcal{W}(t|t_0; \mathcal{V}) \]

\[ \mathcal{W}(t|t_0) = \int_{t_0}^{t} \mathcal{V}(\tau) d\tau + \alpha_1 \int_{t_0}^{t} \left\{ \mathcal{V}(\tau), \int_{t_0}^{\tau} \mathcal{V}(\tau') d\tau' \right\} d\tau + \]

\[ + \alpha_2^2 \int_{t_0}^{t} \left\{ \mathcal{V}(\tau), \int_{t_0}^{\tau} \left\{ \mathcal{V}(\tau'), \int_{t_0}^{\tau'} \mathcal{V}(\tau'') d\tau'' \right\} d\tau' \right\} d\tau + \]

\[ + \alpha_1 \alpha_2 \int_{t_0}^{t} \left\{ \mathcal{V}(\tau), \int_{t_0}^{\tau} \mathcal{V}(\tau') d\tau' \right\}, \int_{t_0}^{\tau} \mathcal{V}(\tau') d\tau' \right\} d\tau + \ldots \]

Here is a commutator for any two operators. Similar formulae can be evaluated up to any order.
One can introduce the following presentation for a new operator

\[ \mathcal{W}_\lambda(t|t_0; \mathcal{V}) = \sum_{k=1}^{\infty} \lambda^k \mathcal{W}_k(t|t_0; \mathcal{V}) \]

After some transformation we can obtain the following family of equalities:

\[ \mathcal{W}_1(t|t_0; \mathcal{V}) = \int_{t_0}^{t} \mathcal{V}(\tau) d\tau, \]

\[ \mathcal{W}_2(t|t_0; \mathcal{V}) = -\frac{1}{2} \int_{t_0}^{t} \int_{t_0}^{\tau} \{ \mathcal{V}(\tau), \mathcal{V}(\tau') \} d\tau' d\tau. \]

\[ \mathcal{W}_3(t|t_0; \mathcal{V}) = \frac{1}{6} \int_{t_0}^{t} \int_{t_0}^{\tau} \int_{t_0}^{\tau'} (\{ \mathcal{V}(\tau), \mathcal{V}(\tau') \}, \mathcal{V}(\tau'')) + \]

\[ + \{ \mathcal{V}(\tau''), \mathcal{V}(\tau') \}, \mathcal{V}(\tau) \} d\tau'' d\tau' d\tau. \text{ and so on.} \]
We can note the (according to the Ado lemma) every finite-dimensional algebra has faithful finite-dimensional representation. This allows us to use matrix algebras Lie. Using the above mentioned presentations we can obtain the following operator estimation for previous series. For example,

\[
\mathcal{W}(t|t_0) = \int_{t_0}^{t} A(\tau) d\tau + \alpha_1 \int_{t_0}^{t} \left\{ A(\tau), \int_{t_0}^{\tau} A(\tau') d\tau' \right\} d\tau + \alpha_2^2 \int_{t_0}^{t} \left\{ A(\tau), \int_{t_0}^{\tau} \left\{ A(\tau'), \int_{t_0}^{\tau'} A(\tau'') d\tau'' \right\} d\tau' \right\} d\tau + \\
+ \alpha_1 \alpha_2 \int_{t_0}^{t} \left\{ \left\{ A(\tau), \int_{t_0}^{\tau} A(\tau') d\tau' \right\}, \int_{t_0}^{\tau} A(\tau') d\tau'' \right\} d\tau + \ldots
\]

Whence it follows

\[
\|\mathcal{W}(t|t_0)\| \leq A(t) \left( 1 + 2|\alpha_1| A(t) + 4A^2(t)C_2 + 8A^3(t)C_3 + \sum_{l \geq 4} (2A(t))^l C_l \right),
\]

where

\[
A(t) = \int_{t_0}^{t} \|A(\tau)\| d\tau
\]
Let be $W = \sum_{k>0} W^k$, where $W^k$ enclose all $k$ nested Lie brackets. Then we have the following inequality $||W^k(t|t_0)|| \leq A(t) (2A(t))^k C_k$, $k \geq 0$ and for coefficients $\alpha_{2k}$ we have

$$|\alpha_{2m}| \leq \frac{2}{(2\pi)^{2m}} \sum_{k \geq 1} \frac{1}{2^{2k}} < \frac{4}{(2\pi)^{2m}}.$$ 

Let be $M = \int_{t_0}^{T_2} A(\tau) d\tau$, then $||W^k||_{L1} \leq 2^k M^{k+1} C^k$ for all sufficiently great $k$. The majorant series with general members $2^k M^{k+1} C_k$ will be converge (according to D’Alembert criterion) if there hold the following inequality

$$\lim_{k \to \infty} \frac{2^{k+1} M^{k+2} C_{k+1}}{2^k M^{k+1} C_k} = q < 1.$$ 

For $k = 2l$,

$$q = 2M \lim_{l \to \infty} \frac{C_{2l+1}}{C_{2l}} = 2M \lim_{l \to \infty} \frac{\alpha_{2l} + C_{3}C_{2l-4}}{\alpha_{2l} + C_{2l-2}C_{2l-4}} = 2M.$$ 

So the majorizing series will converge on the assumption of $M < 1/2$

Therefore our series will converge absolutely.
Mathematical models -3
The convergence problem

We can derive corresponding conditions for convergence of matrix formalism for ODE’s. Let cite corresponding estimations.

Let be \( \frac{dX}{dt} = \sum_{k=1}^{\infty} \mathbb{P}^{1k} X^{[k]} \), from where \( X(t) = \sum_{k=1}^{\infty} \mathbb{R}^{1k}(t|t_0)X_0^{[k]} \) and we have \( \|X_0\| \leq r \), and

\[
\left\| \frac{\partial^k F(X, U, t)}{\partial x_1^{k_1} \ldots x_n^{k_n}} \right\| \leq \varphi(t), \quad M = \int_T \varphi(t)dt, \quad L = \sup_{t, \tau \in T} \|\mathbb{R}^{11}(t, \tau)\|.
\]

We can show that there are the next inequalities

\[
\sup_{t, \tau \in T} \|\mathbb{R}^{jj}(t, \tau)\| \leq jL^j \quad \text{and} \quad \|\mathbb{P}^{jj}(t)\| \leq \frac{\varphi(t)}{(j-1)!}.
\]

Let be \( J_i(L, M) = \left\{ \begin{array}{ll}
\prod_{k=3}^{i} \left\{ \frac{L^{k-1}M(k-1)}{(k-2)!} \right\} + 1, & i \geq 3, \\
1, & i = 2
\end{array} \right. \), then we have (\( \overline{X} \) is an exact solution):

\[
\|\overline{X} - X_N\| \leq \sum_{k=N+1}^{\infty} \frac{r^k L^{k+1} M^k}{(k - 1)!} J_k(L, M).
\]
Mathematical models -3
Preservation of qualitative properties

Usually in beam physics there is used the Hamiltonian formalism for particle beam motion description. This automatically leads us to following equalities

\[ \frac{dX}{dt} = J(X) \frac{\partial \mathcal{H}(X, t)}{\partial X}, \]

where \( J(X) \) is a symplectic matrix \( 2n \times 2n \), for example

\[ J(X) = J_0 = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}. \]

The Jacobi matrix \( M(X, t \mid t_0; \mathcal{M}) = M(X; t \mid t_0) = \frac{\partial M(t \mid t_0; \mathcal{H}) \circ X}{\partial X^*} \) satisfies to the following symplecticity condition

\[ M^*(X; t \mid t_0) J(X) M(X; t \mid t_0) = J(X). \]

Here we have \( \det M(X; t \mid t_0) \equiv 1 \).

According to the matrix formalism one can derive

\[ M(X; t \mid t_0) = \frac{\partial M(t \mid t_0; \mathcal{H}) \circ X}{\partial X^*} = \sum_{k=1}^{\infty} \mathbb{R}^1_k(t \mid t_0; \mathcal{H}) \frac{\partial X^{[k]}}{\partial X^*}. \quad (1) \]

Using the Kronecker product and sum properties we can derive

\[ M(X; t \mid t_0) = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \mathbb{R}^1_k(t \mid t_0; \mathcal{H}) X^{[j]} \otimes E_{2n} \otimes X^{[k-j-1]}. \quad (2) \]
The Preservation of qualitative properties

(qualitative properties)

Replacing (2) into (1) one can derive

\[ M(X; t \mid t_0) = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} R^{1k} (t \mid t_0; \mathcal{H}) X[j] \otimes E \otimes X[k-j-1]. \]

From here we can describe

\[
\left( R^{11} \right)^* J_0 R^{11} + \left( X \otimes E + E \otimes X \right)^* \left( R^{12} \right)^* J_0 R^{11} + \]

\[
+ \left( R^{11} \right)^* J_0 R^{12} (X \otimes E + E \otimes X) + \sum_{k=1, \ l=2}^{\infty} (X^{\odot k})^* (R^{1k})^* J_0 X^{\odot l} = J_0,
\]

were \( X^{\odot (k-1)} = \sum_{j=0}^{k-1} X[j] \otimes E \otimes X[k-j-1]. \) Denoting \( R^{1k} = R^{11} Q^{1k} \) (here \( Q^{11} = E \)) we derive

\[
\sum_{k+l=m} (X^{\odot k})^* (Q^{1(k+1)})^* J_0 Q^{1(l+1)} X^{\odot l} = 0, \quad m \geq 1.
\]

Or the following equalities sequence

\[
(X^{\odot 1})^* (Q^{12})^* J_0 + J_0 Q^{12} X^{\odot 1} = 0,
\]

\[
(X^{\odot 2})^* (Q^{13})^* J_0 + (X^{\odot 1})^* (Q^{12})^* J_0 Q^{12} X^{\odot 1} + J_0 Q^{13} X^{\odot 2} = 0,
\]

\[
(X^{\odot 3})^* (Q^{14})^* J_0 + (X^{\odot 2})^* (Q^{13})^* J_0 Q^{12} X^{\odot 1} + (X^{\odot 1})^* (Q^{12})^* J_0 Q^{13} X^{\odot 2} + J_0 Q^{14} X^{\odot 3} = 0, \quad \ldots
\]
Qualitative properties

The equalities sequences (3) can be solved step-by-step both in analytical and in numerical modes. It should be noted that these equalities impose simple algebraic conditions on corresponding matrix elements. These equalities have the form of linear algebraic equalities with integer coefficients! For example, for four dimensional phase space for the second order matrix $Q^{[2]} = \{q_{ij}\}$ we obtain

$$Q^{12} = \begin{bmatrix}
q_{11} & q_{12} & q_{13} & q_{14} & q_{15} & q_{16} & q_{17} & q_{18} & q_{19} & q_{110} \\
q_{21} & q_{22} & q_{23} & q_{15} & q_{25} & q_{17} & q_{27} & q_{19}/2 & 2 & q_{110} & q_{210} \\
q_{31} & q_{32} & q_{33} & -2q_{11} & -2q_{21} & -q_{12} & -q_{22} & q_{14}/2 & -q_{15} & q_{25}/2 \\
q_{32}/2 & 2 & q_{33} & q_{43} & -q_{12} & -q_{22} & q_{32}/2 & 2 & q_{33} & q_{43} & -q_{12} & -q_{22}
\end{bmatrix}.$$

It should be noted that similar matrices can be precomputed (for example, using Maple or Mathematica packages) and kept that in a special database. This presentation guaranties us fulfilment of the symplecticity conditions up the necessary order for arbitrary interval of independent variable!
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Exact solutions -1

simple example

The correctness of the matrix formalism can be tested for some simple examples.

1. One-dimensional nonlinear equation

\[
\frac{dx}{dt} = K_n x^n.
\]

The exact solution of this equality has the form

\[
x(t) = 1 - n \sqrt[n]{K_n (1 - n) (t - t_0) + x_0^{1-n}}.
\]

For Lie operator we can write

\[
\mathcal{M}(t, t_0) = \exp\{(t - t_0) K_n x^n \frac{\partial}{\partial x}\}.
\]

After some simple calculations one can obtain the desired expression!

\[
\mathcal{M} \circ x_0 = x_0 \sum_{k=0}^{\infty} \left( \frac{1}{1-n} \right) \left( (1-n) K_n (t-t_0) x_0^{n-1} \right)^k = \frac{x_0}{\sqrt[n-1]{1 + (1-n) K_n (t-t_0) x_0^{n-1}}}.
\]
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\[ \frac{dx}{dt} = K_n x^n. \]

The exact solution of this equality has the form

\[ x(t) = \frac{1-n}{n} \sqrt{K_n(1-n)(t-t_0)} + x_0^{1-n}. \]

For Lie operator we can write

\[ \mathcal{M}(t, t_0) = \exp \left\{ (t-t_0) K_n x^n \frac{\partial}{\partial x} \right\}. \]

After some simple calculations one can obtain the desired expression!

\[ \mathcal{M} \circ x_0 = x_0 \sum_{k=0}^{\infty} \binom{1}{1-n} \left( (1-n)K_n(t-t_0)x_0^{n-1} \right)^k = \frac{x_0}{\sqrt[n-1]{1 + (1-n)K_n(t-t_0)x_0^{n-1}}} \]
Exact solutions -1

simple example

The correctness of the matrix formalism can be tested for some simple examples.  

1. One-dimensional nonlinear equation

\[
\frac{dx}{dt} = K_n x^n.
\]

The exact solution of this equality has the form

\[
x(t) = \sqrt[n]{K_n(1-n)(t-t_0)} + x_0^{1-n}.
\]

For Lie operator we can write

\[
\mathcal{M}(t, t_0) = \exp\{(t-t_0)K_n x^n \frac{\partial}{\partial x}\}.
\]

After some simple calculations one can obtain the desired expression!

\[
\mathcal{M} \circ x_0 = x_0 \sum_{k=0}^{\infty} \left( \frac{1}{1-n} \right) \left( (1-n)K_n(t-t_0)x_0^{n-1} \right)^k = \frac{x_0}{n^{-1} \sqrt{1 + (1-n)K_n(t-t_0)x_0^{n-1}}}.
\]
The correctness of the matrix formalism can be tested for some simple examples.

2. Fractionally rational solution of nonlinear equations.

In some cases we search the solution of the nonlinear equations in the fractionally rational form

\[ \mathcal{M} \circ X_0 = X(X_0; t | t_0) = \frac{P_N(X_0; t | t_0)}{Q_L(X_0; t | t_0)}, \]

where

\[ P_N(X_0; t | t_0) = \sum_{k=0}^{N} P^k(t | t_0)X_0^{[k]}, \quad Q_L(X_0; t | t_0) = \sum_{j=0}^{L} (Q_j(t | t_0))^* X_0^{[j]} \]

Let consider the next case \( \mathcal{M} = \mathcal{M}_m = \exp(t - t_0)\mathcal{L}_m \), where \( \mathcal{L}_m = G^*(X_0)\partial/\partial X_0 \).

After some calculation one can obtain

\[ \mathcal{M}_m \circ X_0 = X_0 + \sum_{k=1}^{\infty} \frac{(t - t_0)^k P_{1k}}{k!} X_0^{[k(m-1)+1]}, \quad P_{1k} = \prod_{j=1}^{k} (\mathcal{G}_{m, m}^{\oplus}(j-1)(m-1)+1) \]

Let us introduce

\[ C_m^k = \left((t - t_0)/(k - 1)\right) P_{1k}^{(k-1)} \]
Exact solutions - 2

then

\[ C^k_m = P^k_m - \sum_{j=0}^{L} Q^*_j \otimes C^{k-j}_m, \quad 1 \leq k \leq N, \]

\[ C^k_m + \sum_{j=1}^{k} Q^*_j \otimes C^{k-j}_m = 0, \quad k > N. \]

For the second order nonlinear Hamiltonian equations

\[ \frac{dx}{dt} = ax^2, \quad \frac{dP_x}{dt} = bx^2 - 2axP_x, \quad (4) \]

we can obtain

\[ X = \frac{X_0 + P^2_2 X_0^{[2]} + P^3_2 X_0^{[3]} + P^4_2 X_0^{[4]}}{1 + Q^*_1 X_0}. \quad (5) \]

where

\[ Q_0 = 1, \quad Q_1 = -(t - t_0) \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad P^2_2 = a(t - t_0) \begin{pmatrix} 0 & 0 & 0 \\ -b & 3a & 0 \end{pmatrix}, \]

\[ P^3_2 = a(t - t_0)^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ -b & 3a & 0 & 0 \end{pmatrix}, \quad P^4_2 = \frac{a^2(t - t_0)^3}{3} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -b & 3a & 0 & 0 & 0 \end{pmatrix}. \]

We should note that (5) is exact solution of the equation system (4)!
Energy conservation

It is known that in general cases the symplecticity of the map (exact or approximate map) does not guarantee the energy conservation. That is why we should additionally constrain the used approximated map. In another words on the every step we must guarantee the energy conservation low, which can be written in the following forms

\[ E(Q, P, t_k) = E(Q, P, t_{k-1}), \quad \forall k \geq 1, \quad X = \begin{pmatrix} Q \\ P \end{pmatrix}, \quad \mathcal{M}(t_k | t_{k-1}) \circ E(X, t_k) \equiv E(X, t_{k-1}). \]

These conditions can be realized using some correction procedure. We demonstrate this process using the matrix formalism

\[ \mathcal{M}(t_k | t_{k-1}) \circ E(X, t_k) = E(\mathcal{M}(t_k | t_{k-1}) \circ X, t_{k-1}) = E \left( \sum_{j=1}^{\infty} \mathbb{R}^{[1j]}(t_k | t_{k-1}) X^{[j]} \right) \]

For linear case we have

\[ E(Q_{k-1}, P_{k-1}, t_{k-1}) = \frac{1}{2} \left( X^T(t_{k-1}) \cdot A \cdot X_{k-1}(t_{k-1}) \right), \quad A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \]

\[ E \left( \mathbb{R}^{[11]}(t_k | t_{k-1}) X_{k-1} \right) = \left( X^T_{k-1} \left( \mathbb{R}^{[11]}_k \right)^T \cdot A \cdot \mathbb{R}^{[11]}_k X_{k-1} \right) = \left( X^T_{k-1} \cdot A \cdot X_{k-1} \right) \]

where \( \mathbb{R}^{[11]}_k = \mathbb{R}^{[11]}(t_k | t_{k-1}) \), and we have \( E_k = E_{k-1}! \)
Energy conservation

Similar evaluation for nonlinear Hamiltonian and using full “matrix map”

\[ \sum_{j=1}^{\infty} \mathcal{R}^{[1,j]}(t_k | t_{k-1}) X[j] \]

leads us to the same result. On the practice we apply some truncated transformation of \( N \)-th order

\[ \sum_{j=1}^{N} \mathcal{R}^{[1,j]}(t_k | t_{k-1}) X[j] \]

and similar transformation doesn’t conserve nonlinear Hamiltonian!

**There is a problem:** Can we construct an integration scheme that is both symplectic and energy-conserving properties for a broad class of Hamiltonian systems? The well known Zhang and Marsden theorem answer – in general case – **NO**!

If we want to conserve nonlinear Hamiltonian, than we should “correct a little” our truncated matrix map. In another words, some elements of \( \mathcal{R}^{[1,j]}(t_k | t_{k-1}) \) we should be corrected.

For this purpose we can evaluate some equations (see, an example, the correction procedure for symplectification). Here there are some different approaches. The choice of appropriate variant depends on the practical problem: the symplectification condition is **universal property**, while the energy conservation depends on the energy function (Hamiltonian)!
Conclusion (physical problems - 1)

1. The results are correspond to the existing experimental data.

2. The matrix formalism can be used for different models of beam dynamics (in the frame of the successive approximations approach), including space charge forces.

3. The matrix formalism can be symplectified with comparative ease. Also we can compute approximate invariants for particle beams.

4. The matrix formalism permits different correction procedures for energy conservation.
Conclusion (mathematical problems - 1)

1. The basic principal difference the matrix formalism for presentation of motion equations in the form of ODE’s or Hamiltonian equations: we handle with **two dimensional matrices** instead of multidimensional tensors, similar in MAD, Transport, COSY Infinity and so on.

2. The “improvement” of corresponding models is realized using step-by-step process (using **increasing of approximation order and variation of corresponding matrices**).

3. Linear and “nonlinear” matrices can be evaluated both in **symbolic** (and to keep in special data bases) and in **numerical forms** (using appropriate numerical methods, for example, symplectic Runge-Kutta method or others for corresponding matrix ODE’s).
1. The matrix formalism is **compatible with optimization procedures** of beam dynamics. For this purpose we can use only corresponding matrix elements.

2. The matrix formalism admit **parallelization** and **distribution** procedures (including Grid- and Cloud technologies) naturally.

3. The matrix formalism can be easily embedded into the **Virtual Accelerator** concept.
Some of corresponding papers (in English)


