DYNAMICAL APERTURE BEYOND PERTURBATIONS: FROM QUALITATIVE ANALYSIS TO MAPS

A.N. Fedorova, M.G. Zeitlin*, IPME RAS, St. Peterburg, Russia

Abstract

We start with a qualitative approach based on the detailed analysis of smoothness classes of the underlying functional spaces provided possible evaluation of the dynamical aperture in general nonlinear/polynomial models of particle/beam motion in accelerators. We present the applications of discrete multiresolution analysis technique to the maps which arise as the invariant discretization of continuous nonlinear polynomial problems. It provides a generalization of the machinery of local nonlinear harmonic analysis, which can be applied for both discrete and continuous cases and allows to construct the explicit multiresolution decomposition for solutions of discrete problems which are the correct discretizations of the corresponding continuous cases.

INTRODUCTION

The estimation of the dynamic aperture of accelerators is an important, complicated and long standing problem. From the formal point of view the aperture is some border between two types of dynamics: relative regular and predictable motion along of acceptable orbits or fluxes of orbits corresponding to KAM tori and stochastic motion with particle losses blown away by the Arnold diffusion and/or chaotic motions. According to the standard point of view this transition is being done by some analogues with map technique [1]. Consideration for aperture of n-pole Hamiltonians with kicks

$$H = \frac{p_x^2}{2} + \frac{K_x(s)}{2}x^2 + \frac{p_y^2}{2} + \frac{K_y(s)}{2}y^2 + \frac{1}{3!B\rho}\frac{\partial^2 B_z}{\partial x^2}(x^3 - 3xy^2)L\sum_{k=-\infty}^{\infty}\delta(s - kL) + \ldots \quad (1)$$

is done by linearisation and discretization of canonical transformation and the result resembles (pure formally) standard mapping. This leads, by using the Chirikov criterion of resonance overlapping, to the evaluation of aperture via amplitude of the following global harmonic representation:

$$x^{(n)}(s) = \sqrt{2J_{(n)}\beta_x(s)}\cdot \cos\left(\psi_1 - \frac{2\pi\nu}{L}s + \int_0^s \frac{\mathrm{d}s'}{\beta_x(s')}\right). \quad (2)$$

The goal of this paper is two-fold and presents a sketch of alternative approaches located beyond any linearization or perturbation approaches. In the next part, we consider some qualitative criterion which is based on the attempts of more realistic understanding of the existing difference between motion in KAM region and stochastic regions: motion in KAM regions may be described by regular functions only (without the influence of complicated internal structures leading to nonuniform hyperbolicity generating chaos) while motion in stochastic regions/layers may be described by functions with internal (self-similar, e.g.) structures (definitely, created by actions of symmetry generated groups, like discrete groups, or by actions of hidden symmetries of background functional space, like affine group in the most simple case) i.e. fractal type functions which realized the proper orbits [2]. In the subsequent section according to the invariant Marsden-Veselov approach, we consider symplectic and Lagrangian background for the case of discretization of flows by the corresponding maps [3]. Affter that, in the next section, we present the construction of the corresponding solutions by applications of the multiscale approach of A. Harten [4] based on generalization of multiresolution analysis for the case of maps. Such approaches provide the principles and the possibilities for the control of aperture behaviour in the space of machine parameters. All details, constructions, and results can be found in [5].

QUALITATIVE ANALYSIS

The fractal or chaotic image is a function (distribution) which has structure at all underlying scales. Such objects have additional nontrivial details on any level of resolution. But they cannot be represented by smooth functions, because they resemble constants at small scales [2]. We need to find self-similarity behaviour during movement to small scales for the functions describing non-regular motion. So, if we look on a “fractal” function f (e.g. the Weierstrass function) near an arbitrary point at different scales, we find the same function up to the scaling factor. Consider the fluctuations of such function f near some point x_0

$$f_{loc}(x) = f(x_0 + x) - f(x_0), \quad (3)$$

then we have the renormalization (group)–like behaviour/transformation

$$f_{x_0}(\lambda x) \sim \lambda^{\alpha(x_0)} f_{x_0}(x), \quad (4)$$

where $\alpha(x_0)$ is the so-called local scaling exponent or Hölder exponent of the function f at x_0. According to [2] general functional spaces and scales of spaces can be characterized through wavelet coefficients or wavelet transforms. Let us consider continuous wavelet transform

$$W_g f(b,a) = \int_{R^n} \mathrm{d}x \frac{1}{a^n}\bar{g}\left(\frac{x-b}{a}\right) f(x),$$

* zeitlin@math.ipme.ru

ISBN 978-3-95450-181-6

$b \in R^n$, $a > 0$, w.r.t. analyzing wavelet g, which is strictly admissible, i.e.

$$C_{g,g} = \int_0^\infty \frac{da}{a} |\hat{g}(\bar{a}\, k)|^2 < \infty.$$

Wavelet transform has the following covariance property under action of the underlying affine group:

$$W_g(\lambda a, x_0 + \lambda b) \sim \lambda^{\alpha(x_0)} W_g(a, x_0 + b). \qquad (5)$$

So, if the Hölder exponent of (distribution) $f(x)$ around the point $x = x_0$ is $h(x_0) \in (n, n+1)$, then we have the following behaviour of $f(x)$ around $x = x_0$:

$$f(x) = c_0 + c_1(x - x_0) + \ldots + c_n(x - x_0)^n + c|x - x_0|^{h(x_0)}.$$

Let the analyzing wavelet has n_1 ($> n$) vanishing moments, then

$$W_g(f)(x_0, a) = C a^{h(x_0)} W_g(f)(x_0, a) \qquad (6)$$

and

$$W_g(f)(x_0, a) \sim a^{h(x_0)},$$

when $a \to 0$. But if $f \in C^\infty$ at least in the point x_0, then

$$W_g(f)(x_0, a) \sim a^{n_1},$$

when $a \to 0$ [2]. This shows that the localization of wavelet coefficients at small scale is linked to local regularity. As a rule, the faster the wavelet coefficients decay, the more the analyzed function is regular. So, transition from regular motion to chaotic one may be characterised as the changing of the Hölder/scaling exponent of the function which describes motion. This gives a criterion of the appearance of fractal behaviour and may determine, at least in principle, the dynamic aperture as well as the dependence on parameters of the type of behaviour.

INVARIANT DISCRETIZATION

Discrete variational principles lead to evolution dynamics analogous to the Euler-Lagrange equations [3]. Let Q be a configuration space, then a discrete Lagrangian is a map $L : Q \times Q \to \mathbf{R}$, L is obtained by approximating the given Lagrangian. For $N \in N_+$ the action sum is the map $S : Q^{N+1} \to \mathbf{R}$ defined by

$$S = \sum_{k=0}^{N-1} L(q_{k+1}, q_k), \qquad (7)$$

where $q_k \in Q$, $k \geq 0$. The action sum is the discrete analog of the action integral in continuous case. Extremizing S over $q_1, ..., q_{N-1}$ with fixing q_0, q_N we have the discrete Euler-Lagrange equations (DEL):

$$D_2 L(q_{k+1}, q_k) + D_1(q_k, q_{q-1}) = 0, \qquad (8)$$

for $k = 1, ..., N - 1$.

Let

$$\Phi : Q \times Q \to Q \times Q \qquad (9)$$

and

$$\Phi(q_k, q_{k-1}) = (q_{k+1}, q_k) \qquad (10)$$

is a discrete function (map), then we have for DEL:

$$D_2 L \circ \Phi + D_1 L = 0 \qquad (11)$$

or in coordinates q^i on Q we have DEL

$$\frac{\partial L}{\partial q_k^i} \circ \Phi(q_{k+1}, q_k) + \frac{\partial L}{\partial q_{k+1}^i}(q_{k+1}, q_k) = 0. \qquad (12)$$

It is very important that the map Φ exactly preserves the discretization of the symplectic form ω [3]:

$$\omega = \frac{\partial^2 L}{\partial q_k^i \partial q_{k+1}^j}(q_{k+1}, q_k) dq_k^i \wedge dq_{k+1}^j \qquad (13)$$

MAPS: MULTIRESOLUTION

Our approach to solutions of equations (12) is based on applications of general and very efficient methods developed by A. Harten [3], who produced a "General Framework" for multiresolution representation of discrete data. It is based on consideration of basic operators, decimation and prediction, which connect adjacent resolution levels. These operators are constructed from two basic blocks: the discretization and reconstruction operators. The former obtains discrete information from a given continuous functions (flows), and the latter produces an approximation to those functions, from discrete values, in the same function space to which the original function belongs. A "new scale" is defined as the information on a given resolution level which cannot be predicted from discrete information at lower levels. If the discretization and reconstruction are local operators, the concept of "new scale" is also local. The scale coefficients are directly related to the prediction errors, and thus to the reconstruction procedure. If scale coefficients are small at a certain location on a given scale, it means that the reconstruction procedure on that scale gives a proper approximation of the original function at that particular location. This approach may be considered as some generalization of standard wavelet analysis approach. It allows to consider multiresolution decomposition when usual approach is impossible (singular, non-regular behaviour). We demonstrated the discretization of kick/Dirac function by wavelet packets on Fig. 1 and Fig. 2.

Let F be a linear space of mappings

$$F \subset \{f | f : X \to Y\}, \qquad (14)$$

where X, Y are linear spaces. Let also D_k be a linear operator

$$D_k : f \to \{v^k\}, \quad v^k = D_k f, \; v^k = \{v_i^k\}, \quad v_i^k \in Y. \qquad (15)$$

This sequence corresponds to k level discretization of X. Let

$$D_k(F) = V^k = \mathrm{span}\{\eta_i^k\} \qquad (16)$$

ISBN 978-3-95450-181-6

and the coordinates of $v^k \in V^k$ in this basis are $\hat{v}^k = \{\hat{v}^k_i\}$, $\hat{v}^k \in S^k$:

$$v^k = \sum_i \hat{v}^k_i \eta^k_i, \qquad (17)$$

D_k is a discretization operator. Main goal is to design a multiresolution scheme (MR) [4] that applies to all sequences $s \in S^L$, but corresponds for those sequences $\hat{v}^L \in S^L$, which are obtained by the discretization (14).

Since D_k maps F onto V^k then for any $v^k \subset V^k$ there is at least one f in F such that $D_k f = v^k$. Such correspondence from $f \in F$ to $v^k \in V^k$ is reconstruction and the corresponding operator is the reconstruction operator R_k:

$$R_k : V_k \to F, \qquad D_k R_k = I_k, \qquad (18)$$

where I_k is the identity operator in V^k (R^k is right inverse of D^k in V^k).

Given a sequence of discretization $\{D_k\}$ and sequence of the corresponding reconstruction operators $\{R_k\}$, we define the operators D^{k-1}_k and P^k_{k-1}

$$D^{k-1}_k = D_{k-1} R_k : V_k \to V_{k-1} \qquad (19)$$
$$P^k_{k-1} = D_k R_{k-1} : V_{k-1} \to V_k$$

If the set D_k is nested [4], then

$$D^{k-1}_k P^k_{k-1} = I_{k-1} \qquad (20)$$

and we have for any $f \in F$ and any $p \in F$ for which the reconstruction R_{k-1} is exact:

$$D^{k-1}_k (D_k f) = D_{k-1} f, \quad P^k_{k-1}(D_{k-1} p) = D_k p \qquad (21)$$

Let us consider any $v^L \in V^L$, Then there is $f \in F$ such that

$$v^L = D_L f, \qquad (22)$$

and it follows from (21) that the process of successive decimation [4]

$$v^{k-1} = D^{k-1}_k v^k, \qquad k = L, ..., 1 \qquad (23)$$

yields for all k

$$v^k = D_k f \qquad (24)$$

Thus the problem of prediction, which is associated with the corresponding MR scheme, can be stated as a problem of approximation: knowing $D_{k-1} f$, $f \in F$, find a "good approximation" for $D_k f$. It is very important that each space V^L has a multiresolution basis

$$\bar{B}_M = \{\bar{\phi}^{0,L}_i\}_i, \{\{\bar{\psi}^{k,L}_j\}_j\}^L_{k=1} \qquad (25)$$

and that any $v^L \in V^L$ can be written as

$$v^L = \sum_i \hat{v}^0_i \bar{\phi}^{0,L}_i + \sum_{k=1}^{L} \sum_j d^k_j \bar{\psi}^{k,L}_j, \qquad (26)$$

where $\{d^k_j\}$ are the k scale coefficients of the associated MR, $\{\hat{v}^0_i\}$ is defined by (17) with $k = 0$. If $\{D_k\}$ is a nested sequence of discretization [4] and $\{R_k\}$ is any corresponding sequence of linear reconstruction operators, then we have from (26) for $v^L = D_L f$ applying R_L:

$$R_L D_L f = \sum_i \hat{f}^0_i \phi^{0,L}_i + \sum_{k=1}^{L} \sum_j d^k_j \psi^{k,L}_j, \qquad (27)$$

$$\phi^{0,L}_i \in F, \qquad (28)$$
$$\psi^{k,L}_j = R_L \bar{\psi}^{k,L}_j \in F, D_0 f = \sum \hat{f}^0_i \eta^0_i.$$

When $L \to \infty$ we have sufficient conditions which ensure that the limiting process in (27, 28) yields a multiresolution basis for F (14). Then, according to (25), (26) we have a very useful representation via the multiscale form for solutions (26) of map version (12) of initial equations obtained from (1) as well as for various maps constructions which are discrete counterparts for continuous cases considered in [5].

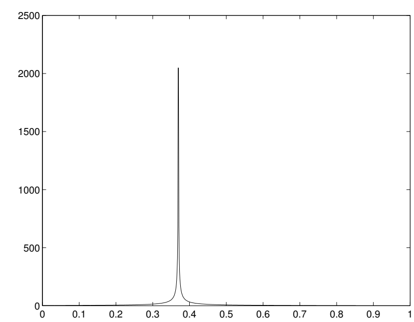

Figure 1: Kick/Delta function.

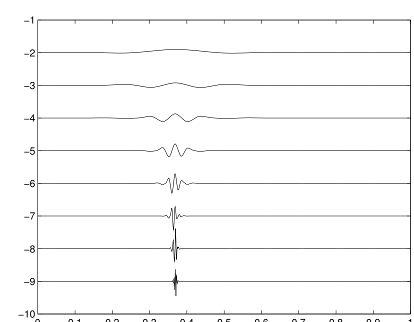

Figure 2: Discretization via wavelet packets.

REFERENCES

[1] A. Chao, *Handbook of Accelerator Physics and Engineering*, World Scientific, 1999.

[2] A. Arneodo, *Wavelets*, p. 349, Oxford, 1996; M. Holschneider, *Wavelets*, Clarendon, 1998.

[3] J.E. Marsden, *Park City Lectures on Mechanics, Dynamics and Symmetry*, Caltech, 1998.

[4] A. Harten, *SIAM J. Numer. Anal.*, 31, 1191-1218, 1994.

[5] Antonina N. Fedorova, Michael G. Zeitlin, papers/preprints at `http://math.ipme.ru/zeitlin.html`; “Orbital motion in multipole fields via multiscale decomposition”, this Volume.

ISBN 978-3-95450-181-6